FACULTY OF ARTS AND SCIENCE

University of Toronto

FINAL EXAMINATIONS, APRIL/MAY 2002

MAT 133Y1Y

Calculus and Linear Algebra for Commerce

PART A. MULTIPLE CHOICE

1. *[3 marks]*

$$\lim_{x \to -8^-} \frac{x+8}{|64 - x^2|}$$

Ⓐ equals 1

Ⓑ equals 0

Ⓒ equals $\frac{1}{16}$

Ⓓ equals $-\frac{1}{16}$

Ⓔ does not exist

2. *[3 marks]*

The system of linear equations

$$\begin{aligned} 2x_1 - 4x_2 - 6x_3 &= 2 \\ x_1 - 2x_2 - x_3 &= 1 \\ -x_1 + 2x_2 + 5x_3 &= -1 \end{aligned}$$

has

Ⓐ no solution

Ⓑ unique solution with $x_3 = 0$

Ⓒ infinitely many solutions with $x_2 = 0$

Ⓓ infinitely many solutions with $x_3 = 0$

Ⓔ infinitely many solutions with $x_1 = 0$

3. *[3 marks]*

The equation $(xy)^3 = 4\sqrt{y} - 3$ defines y implicitly as a function of x near the point $x = 1$, $y = 1$. At this point y' is

Ⓐ undefined

Ⓑ 3

Ⓒ 0

Ⓓ -1

Ⓔ -3

4. *[3 marks]*

If $f(x) = \dfrac{x}{x^2+1}$, then

Ⓐ f has an absolute maximum and an absolute minimum

Ⓑ f has an absolute maximum but only a local minimum

Ⓒ f has an absolute minimum but only a local maximum

Ⓓ f has no absolute maximum and no absolute minimum, but only a local maximum and a local minimum

Ⓔ f has no absolute or local extrema at all

5. *[3 marks]*

If the total revenue function is $r(q) = \dfrac{2500q}{\ln(10q+10)}$, where q is the number of units of some commodity, then the marginal revenue when $q = 100$ is closest to

Ⓐ 310

Ⓑ 36,139

Ⓒ 356

Ⓓ 248

Ⓔ 2,475

6. *[3 marks]*

$$\lim_{x \to 1} \frac{2^x - 2}{\ln x}$$

Ⓐ $= 2 \ln 2$

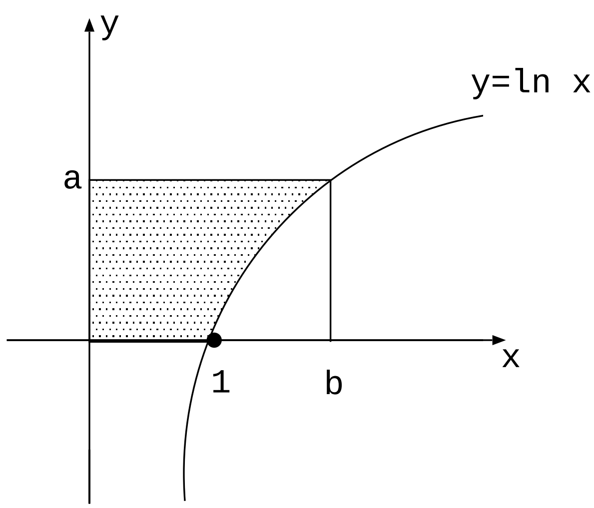
Ⓑ $= 2$

Ⓒ $= 0$

Ⓓ $= \dfrac{1}{2}$

Ⓔ does not exist

7. *[3 marks]*



The area of the shaded region is given by:

Ⓐ $\displaystyle\int_0^a \ln y \, dy$

Ⓑ $\displaystyle\int_0^b (a - \ln x) \, dx$

Ⓒ $\displaystyle\int_0^a e^y \, dy$

Ⓓ $\displaystyle\int_0^a (b - e^y) \, dy$

Ⓔ $\displaystyle\int_0^b \ln x \, dx - \int_1^b \ln x \, dx$

8. *[3 marks]*

If A and B are real numbers such that $\dfrac{x+9}{x^2-9} = \dfrac{A}{x+3} + \dfrac{B}{x-3}$ (wherever both sides of the equality are defined) then $A =$

Ⓐ -2

Ⓑ 0

Ⓒ 1

Ⓓ -1

Ⓔ 2

9. *[3 marks]*

$\displaystyle\int_0^1 xe^{3x}\,dx =$

Ⓐ $\dfrac{1}{3}$

Ⓑ $\dfrac{1+2e^3}{9}$

Ⓒ $\dfrac{-1+4e^3}{9}$

Ⓓ $\dfrac{1+2e^3}{3}$

Ⓔ $\dfrac{-1-4e^3}{9}$

10. *[3 marks]*

$\displaystyle\int_2^\infty \frac{dx}{\sqrt{x}}$

Ⓐ equals 2

Ⓑ diverges

Ⓒ equals 1

Ⓓ equals $2\sqrt{2}$

Ⓔ equals $\sqrt{2}$

11. *[3 marks]*

From time $t = 0$ until time $t = 8$ years, cash flows into an account which earns interest continuously at a rate of 5% per year. If, at time t years, the rate of the cash flow is $1000e^{.3t}$ dollars per year, then the present value of the total cash flow (to the nearest dollar) is

Ⓐ \$25,556

Ⓑ \$27,038

Ⓒ \$21,297

Ⓓ \$18,254

Ⓔ \$33,411

12. *[3 marks]*

If y is a function of x such that $e^y y' = 2x$ and $y = 1$ when $x = 0$, then when $x = 1$, $y =$

Ⓐ 1

Ⓑ 0

Ⓒ $\ln(1+e)$

Ⓓ $\ln 3$

Ⓔ $-\ln 2$

13. *[3 marks]*

If $f(x,y) = x^y$ for $x > 0$, $y > 0$, then $\dfrac{\partial f}{\partial x}(4, \frac{1}{2}) + \dfrac{\partial f}{\partial y}(4, \frac{1}{2}) =$

Ⓐ 0

Ⓑ $\dfrac{1}{2} - \dfrac{\ln 2}{16}$

Ⓒ $\dfrac{1}{2}$

Ⓓ 2

Ⓔ $\dfrac{1}{4} + 2\ln 4$

14. *[3 marks]*

Suppose a manufacturer of two related products A and B has a total cost function

$$C(q_A, q_B) = 0.2q_A^2 + 0.1q_Aq_B + 10q_B$$

When $q_A = 100$ and $q_B = 50$ it is known that $\dfrac{\partial q_A}{\partial p_A} = -1$ and $\dfrac{\partial q_B}{\partial p_A} = 2$. The rate of change of total cost with respect to p_A when $q_A = 100$ and $q_B = 50$ is

Ⓐ 0

Ⓑ 5

Ⓒ -5

Ⓓ 15

Ⓔ -15

15. *[3 marks]*

$\displaystyle\int_0^1 \int_1^2 (x^2 + y^3)\, dx\, dy$ is closest to

Ⓐ 1.6

Ⓑ 2.6

Ⓒ 3.6

Ⓓ 4.6

Ⓔ 5.6

PART B. WRITTEN-ANSWER QUESTIONS

B1. *[13 marks]*

In January, 1991, Jane's employer gave her the option of continuing full-time work until any month, to be chosen by Jane, after which she would work part-time (for a much smaller salary). Accordingly, at the end of June, 1991 and every 6 months thereafter, she made equal deposits into an account earning 6% interest compounded monthly, until her last deposit at the end of December, 2000.

(Suggestion: Read both part-questions before working on either).

(a) Jane's plan was to begin part-time work in 2001 and to withdraw \$5,000 from the account every 6 months for 10 years: the first withdrawal at the end of June, 2001, and the last at the end of December, 2010. What should the amount of each deposit be, in order to achieve this goal? You may assume the account's interest will continue at 6% compounded monthly.

(b) Just after Jane's last deposit, she changed her mind about working part-time and immediately invested the money she had accumulated in bonds of face value \$1,000. Each bond had a semiannual yield rate of 4% and 20 semiannual coupons worth \$30 each — the first coupon redeemable at the end of June, 2001 and the last coupon redeemable when the bond matures at the end of December, 2010. How many bonds was Jane able to buy with her savings?

B2. *[10 marks]*

(a) Find $\displaystyle\int e^{\sqrt{x}}\,dx$

(Hint: let $u = \sqrt{x}$)

(b) Find the total area of the region between the graphs of $y = e^{\sqrt{x}}$ and $y = e$ from

B3. *[12 marks]*

[5] (a) Suppose $f(x)$ is continuous on $[-4, 4]$ and that

$$\begin{aligned} f(-4) &= 1 \\ f(-2) &= 2 \\ f(0) &= 5 \\ f(2) &= 3 \\ f(4) &= 2 \end{aligned}$$

Use the trapezoidal rule with $n = 4$ to approximate $\displaystyle\int_{-4}^{4} f(x)\,dx$.

[7] (b) Evaluate $\displaystyle\int_{1}^{\infty} \frac{x-1}{x^2(x+1)}\,dx$

B4. *[10 marks]*

Find and classify the critical points of $w = xy(20 - x - 2y)$.

B5. *[10 marks]*

Use the method of **Lagrange multipliers** (no other method will be given any marks) to find the critical points of the function $V = x^2y^2$ subject to the constraint $2x + 4y = 40$. Which of these critical points gives the largest value of V for $x \geq 0$, $y \geq 0$?

Solutions to April 2002 Exam, MAT133Y

PART A

1. ANSWER: (D)

$$x < -8 \qquad x^2 > 64$$

$$\frac{x+8}{|64-x^2|} = \frac{x+8}{x^2-64} = \frac{1}{x-8} \to \boxed{-\tfrac{1}{16} \quad \text{as} \quad x \to -8^-}$$

2. ANSWER: (D)

$$\left(\begin{array}{rrr|r} 2 & -4 & -6 & 2 \\ 1 & -2 & -1 & 1 \\ -1 & -2 & 5 & -1 \end{array}\right) \longrightarrow \left(\begin{array}{rrr|r} 1 & -2 & -3 & 1 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 2 & 0 \end{array}\right) \longrightarrow \left(\begin{array}{rrr|r} 1 & -2 & -3 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

Line 2 says $x_3 = 0$

Line 1 then says $x_1 - 2x_2 = 1$

$\boxed{\text{So, infinity many solutions with } x_3 = 0}$

3. ANSWER: (E)

$$3(xy)^2(y + xy') = 2y^{-\frac{1}{2}}y'$$

$$3(1 + y') = 2y' \text{ at } (1,1)$$

$$\boxed{y' = -3}$$

4. ANSWER: (A)

$$f'(x) = 0$$

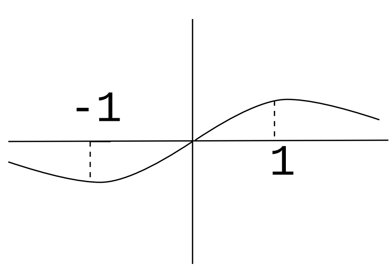
$$f'(x) = \frac{(x^2+1) - 2x^2}{(x^2+1)^2} = \frac{1-x^2}{(x^2+1)} \qquad x = 1 \quad \text{or} \quad x = -1$$

$$\lim_{x \to \pm\infty} f(x) = 0$$

$$(-\infty, -1) \ f \text{ decreases}$$

$$(-1, 1) \ f \text{ increases}$$

$$(1, \infty) \ f \text{ decreases}$$



$\boxed{f \text{ has an absolute max and an absolute min.}}$

5. ANSWER: Ⓐ

$$\frac{dr}{dq} = \frac{[\,\ln(10q+10) - \dfrac{q}{10q+10}\cdot 10\,]\,2500}{[\,\ln(10q+10)\,]^2} = \frac{[\,\ln 1010 - \dfrac{1000}{1010}\,]\,2500}{(\ln 1010)^2}$$

$$\approx 309.67 \approx \boxed{310}$$

6. ANSWER: Ⓐ

$$\frac{0}{0} : \lim_{x\to 1} \frac{2^x - 2}{\ln x} = \lim_{x\to 1} \frac{2^x \ln 2}{\frac{1}{x}} = \boxed{2\ln 2}$$

7. ANSWER: Ⓒ

$$\int_0^a x\; dy = \boxed{\int_0^a e^y\; dy}$$

8. ANSWER: Ⓓ

$$A(x-3) + B(x+3) = x + 9$$

$$x = -3 \Rightarrow -6A = 6$$

$$\boxed{A = -1}$$

9. ANSWER: Ⓑ

$$u = x \qquad dv = e^{3x}\; dx$$

$$du = dx \qquad v = \frac{e^{3x}}{3}$$

$$\begin{aligned}
\int_0^1 xe^{3x}\; dx &= \frac{xe^{3x}}{3}\Big|_0^1 - \frac{1}{3}\int_0^1 e^{3x}\; dx \\
&= \frac{e^3}{3} - \frac{1}{3}\,\frac{e^{3x}}{3}\Big|_0^1 \\
&= \frac{e^3}{3} - \frac{e^3}{9} + \frac{1}{9} \\
&= \boxed{\frac{2e^3+1}{9}}
\end{aligned}$$

10. ANSWER: Ⓑ

$$\begin{aligned}
\lim_{R\to\infty} \int_2^R \frac{dx}{\sqrt{x}} &= \lim_{R\to\infty} 2\sqrt{x}\Big|_2^R \\
&= \lim_{R\to\infty} (2\sqrt{R} - 2\sqrt{2})
\end{aligned}$$

but $\sqrt{R} \to \infty$ so $\boxed{\text{diverges}}$

11. ANSWER: Ⓐ

$$\begin{aligned}\text{P.V.} &= \int_0^8 1000e^{.3t}e^{-.05t}\ dt\\ &= 1000\int_0^8 e^{.25t}\ dt\\ &= \frac{1000}{.25}e^{.25t}\Big|_0^8\\ &= 4000(e^2-1) \approx 25,556.22\end{aligned}$$

$$\boxed{\approx 25,556}$$

12. ANSWER: Ⓒ

$$\begin{aligned}\int e^y\ dy &= \int 2x\ dx\\ e^y &= x^2 + C\\ e &= C\\ e^y &= x^2 + e\\ \text{when } x = 1,\ e^y &= 1 + e\end{aligned}$$

$$\boxed{y = \ln(1+e)}$$

13. ANSWER: Ⓔ

$$\frac{\partial f}{\partial x} = yx^{y-1} = \frac{1}{2}\ 4^{-\frac{1}{2}} = \frac{1}{4}$$

$$\frac{\partial f}{\partial y} = x^y \ln x = 4^{\frac{1}{2}} \ln 4 = 2\ln 4$$

Adding, get $\boxed{\frac{1}{4} + 2\ln 4}$

14. ANSWER: Ⓒ

$$\begin{aligned}\frac{\partial C}{\partial p_A} &= \frac{\partial C}{\partial q_A}\frac{\partial q_A}{\partial p_A} + \frac{\partial C}{\partial q_B}\frac{\partial q_B}{\partial p_A}\\ &= (.4q_A + .1q_B)\frac{\partial q_A}{\partial p_A} + (.1q_A + 10)\frac{\partial q_B}{\partial p_A}\\ &= (40+5)(-1) + (10+10)2\\ &= \boxed{-5}\end{aligned}$$

15. ANSWER: Ⓑ

$$\int_0^1 \left[\frac{x^3}{3} + y^3 x\right]_{x=1}^{x=2} dy$$

$$= \int_0^1 \left(\frac{8}{3} - \frac{1}{3}\right) + (2y^3 - y^3)\ dy$$

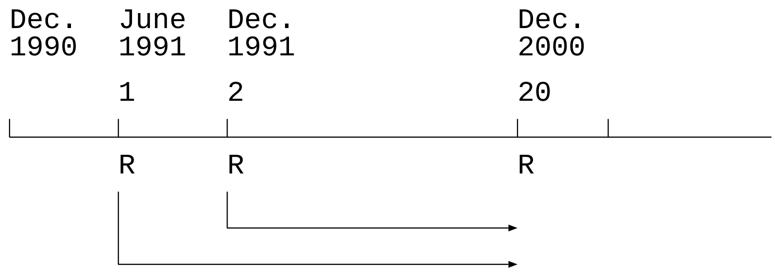
$$= \int_0^1 \left(\frac{7}{3} + y^3\right)\ dy$$

$$= \left[\frac{7}{3}y + \frac{y^4}{4}\right]_0^1 = \frac{7}{3} + \frac{1}{4} \approx 2.58$$

$$\approx \boxed{2.6}$$

PART B

B1.



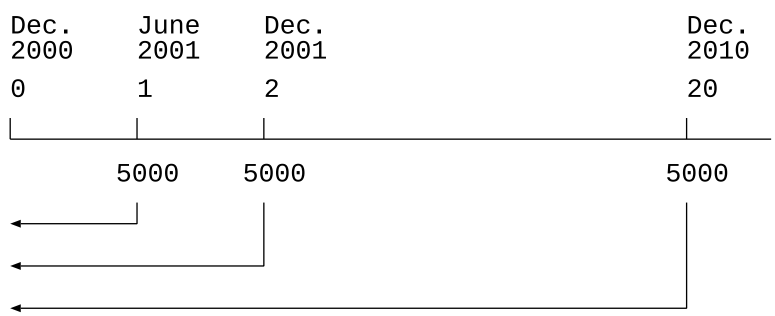
At the end of December 2000, Jane will have accumulated

$$Rs_{\overline{20}|i} = R\frac{(1+i)^{20}-1}{i} = \frac{R[(1.005)^{120}-1]}{(1.005)^6-1}$$

Where

$$(1+i)^2 = \left(1+\frac{.06}{12}\right)^{12}$$
$$1+i = (1.005)^6$$

(a)



$$5,000a_{\overline{20}|i} = Rs_{\overline{20}|i} \qquad 5,000[1-(1+i)^{-20}] = R[(1+i)^{20}-1]$$

$$R = 5,000\frac{[1-(1.005)^{120}]}{[(1.005)^{120}-1]} \approx \boxed{2{,}748.16}$$

(b)

She had accumulatd $Rs_{\overline{20}|i} = 74,128.31$

Bond price $= 1,000(1.04)^{-20} + 30a_{\overline{20}|.04} = \$864.10$

She could buy $\dfrac{74,128.31}{864.10} \approx 85.79$

Which means she could buy $\boxed{\text{85 bonds}}$

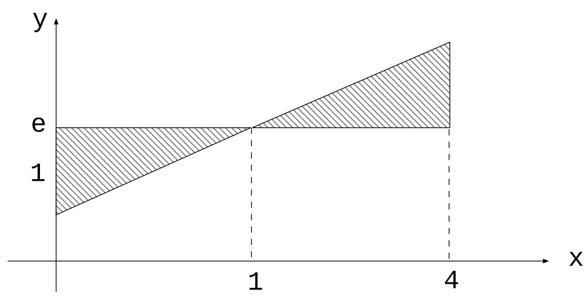
B2.

(a) Let $u = \sqrt{x} \qquad du = \dfrac{1}{2\sqrt{x}}\,dx = \dfrac{1}{2u}\,dx \qquad dx = 2u\,du$

$$\begin{aligned}\int e^{\sqrt{x}}\,dx &= 2\int e^u u\,du \\ &= 2\Big[ue^u - \int e^u\,du\Big] \qquad \text{Let } w = u \qquad dv = e^u du \\ &\qquad\qquad\qquad\qquad\quad dw = du \qquad v = e^u \\ &= 2\big[ue^u - e^u\big] + C \\ &= \boxed{2e^{\sqrt{x}}(\sqrt{x}-1)} + C\end{aligned}$$

(b)

$e^{\sqrt{x}} = e$ when $x = 1$



$$\begin{aligned}A &= \int_0^1 (e - e^{\sqrt{x}}\,dx + \int_1^4 (e^{\sqrt{x}} - x)\,dx \\ &= e - \big[2e^{\sqrt{x}}(\sqrt{x}-1)\big]_0^1 + \big[2e^{\sqrt{x}}(\sqrt{x}-1)\big]_1^4 - 3e \\ &= e - \big[-(-2)\big] + 2e^2(2-1) - 3e \\ &= -2 - 2e + 2e^2 = \boxed{2(e^2 - e - 1)}\end{aligned}$$

B3.

(a)

$$\Delta x = \frac{8}{4} = 2$$

x_0	x_1	x_2	x_3	x_4
-4	-2	0	2	4

$$\begin{aligned}T_4 &= \frac{\Delta x}{2}(y_0 + 2y_1 + 2y_2 + 2y_3 + y_4) \\ &= (1 + 4 + 10 + 6 + 2) = \boxed{23}\end{aligned}$$

(b)

$$\frac{x-1}{x^2(x+1)} = \frac{A}{x} + \frac{B}{x^2} + \frac{C}{x+1}$$

$$Ax(x+1) + B(x+1) + Cx^2 = x - 1$$

$$x = 0 \Rightarrow B = -1$$

$$x = -1 \Rightarrow C = -2$$

$$x = 1 \Rightarrow 2A + 2B + C = 0$$

$$2A - 2 - 2 = 0$$

$$\Rightarrow A = 2$$

$$\int_1^\infty \frac{x-1}{x^2(x+1)}\, dx = \lim_{R\to\infty} \int_1^R (\frac{2}{x} - \frac{1}{x^2} - \frac{2}{x+1}\, dx$$

$$= \lim_{R\to\infty} \left[2\ln x - 2\ln(x+1) + \frac{1}{x}\right]_1^R$$

$$= \lim_{R\to\infty} (2\ln\frac{R}{R+1} + \frac{1}{R}) - (-2\ln 2 + 1)$$

but $\frac{R}{R+1} \to 1$so $\ln\frac{R}{R+1} \to 0$ and $\frac{1}{R} \to 0$

so get $\boxed{2\ln 2 - 1}$

B4.

$$w = 20xy - x^2y - 2xy^2$$

$$\frac{\partial w}{\partial x} = 20y - 2xy - 2y^2 = 2y(10 - x - y) = 0$$

$$\frac{\partial w}{\partial y} = 20x - x^2 - 4xy = x(20 - x - 4y) = 0$$

If $y = 0,\ x(20-x) = 0 \ \Rightarrow x = 0$ or $x = 20$ so critical points at $\boxed{(0,0)}$ and $\boxed{(20,0)}$

If

$$y \neq 0, \quad x + y = 10$$

and either

$$x = 0 \quad \text{so} \quad y = 10$$

$$\boxed{(0,10)}$$

or

$$x \neq 0$$

and

$$x + 4y = 20$$

$$x + y = 10$$

$$3y = 10 \quad y = \frac{10}{3} \quad x = \frac{20}{3}$$

$$\boxed{\left(\frac{20}{3}, \frac{10}{3}\right)}$$

4 critical points

$$\frac{\partial^2 w}{\partial x^2} = -2y \quad \frac{\partial^2 w}{\partial y^2} = -4x \quad \frac{\partial^2 w}{\partial x \partial y} = 20 - 4y - 2x$$

$$D = 4xy - (20 - 4y - 2x)^2$$

$$D(0,0) = -400 < 0 \quad \text{saddle pt}$$

$$D(20,0) = -400 < 0 \quad \text{saddle pt}$$

$$D(0,10) = -400 < 0 \quad \text{saddle pt}$$

$$D\left(\frac{20}{3}, \frac{10}{3}\right) = \frac{4 \times 200}{9} - (20 - \frac{40}{3} - \frac{40}{3})^2$$

$$= \frac{800}{9} - \frac{400}{9} = \frac{400}{9} > 0 \text{ extreme point.}$$

$$w_{xx}\left(\frac{20}{3}, \frac{10}{3}\right) = -\frac{20}{3} < 0 \text{ so } \boxed{\text{local max at } \left(\frac{20}{3}, \frac{10}{3}\right)}$$

B5.

$$\mathcal{L} = x^2y^2 - \lambda(2x + 4y - 40)$$

$$\mathcal{L}_x = 2xy^2 - 2\lambda = 0$$

$$\mathcal{L}_y = 2x^2y - 4\lambda = 0$$

$$\mathcal{L}_\lambda = -(2x + 4y - 40) = 0$$

$$2xy^2 = 2\lambda \qquad x^2y = 2\lambda \qquad \text{so} \qquad x^2y = 2xy^2 \qquad xy(x - 2y) = 0$$

$$x = 0 \Rightarrow \lambda = 0 \quad \text{and} \quad 4y - 40 = 0 \quad \text{so} \quad y = 10 \quad \boxed{\text{critical pt } (0, 10)}$$

$$y = 0 \Rightarrow \lambda = 0 \quad \text{and} \quad 2x - 40 = 0 \quad \text{so} \quad x = 20 \quad \boxed{\text{critical pt } (20, 0)}$$

$$x = 2y \Rightarrow 8y - 40 = 0 \quad \text{so} \quad y = 5,\ x = 10 \quad \text{and} \quad \lambda = 250 \quad \boxed{\text{critical pt } (10, 5)}$$

$V(0, 10) = 0 = V(20, 0)$
but $V(10, 5) = 2,500$ so this gives the largest value when $x \geq 0$ and $y \geq 0$.