FACULTY OF ARTS AND SCIENCE

University of Toronto

FINAL EXAMINATIONS, APRIL/MAY 2001

MAT 133Y1Y

Calculus and Linear Algebra for Commerce

PART A. MULTIPLE CHOICE

1. *[3 marks]*

If Mr. Smith borrows \$5000 from Ms. Jones and repays the loan 125 days later by giving her \$5100, what is the effective annual interest rate of the loan, to the nearest 0.01%? (Note: 1 year = 365 days)

Ⓐ 2.00%

Ⓑ 6.13%

Ⓒ 4.88%

Ⓓ 5.95%

Ⓔ 5.84%

2. *[3 marks]*

Let $A = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ -1 & 1 & 0 \end{bmatrix}$. If $X = A - A^{-1}$, then the entry in the first row and the second column of X is

Ⓐ 2

Ⓑ -3

Ⓒ 3

Ⓓ -1

Ⓔ 0

3. *[3 marks]*

For how many values of k does the system

$$\begin{array}{rcrcrcl} x & + & 2y & - & 3z & = & 4 \\ 3x & - & y & + & 5z & = & 2 \\ 4x & + & y & + & (k^2-14)z & = & k+2 \end{array}$$

have no solution?

Ⓐ for no values of k

Ⓑ for two values of k

Ⓒ for one value of k

Ⓓ for three values of k

Ⓔ for infinitely many values of k

4. *[3 marks]*

Given $f(x) = \begin{cases} 2^{\frac{1}{x+1}} + a & \text{if } x < -1 \\ \dfrac{x^2-1}{x-1} & \text{if } -1 \le x < 1 \\ b - x & \text{if } x \ge 1 \end{cases}$

If f is continuous at $x = \pm 1$, then

Ⓐ $a = -1$ and $b = -1$

Ⓑ $a = 0$ and $b = 3$

Ⓒ $a = 0$ and $b = 1$

Ⓓ $a = 2$ and $b = 3$

Ⓔ $a = 0$ and $b = 2$

5. *[3 marks]*

$\lim\limits_{x \to 0} \dfrac{\ln(1-x)}{e^x - 1}$

Ⓐ does not exist

Ⓑ equals zero

Ⓒ equals 1

Ⓓ equals -1

Ⓔ equals $\sqrt{2}$

6. *[3 marks]*

The average value of $f(x) = x^9$ on the interval $[0, 10]$ is

Ⓐ 8^9

Ⓑ 10^9

Ⓒ 5^9

Ⓓ 9^8

Ⓔ 10^8

7. *[3 marks]*

$$\int 2x \ln(x+1)\, dx =$$

Ⓐ $2x\big[(x+1)\ln(x+1) - (x+1)\big] + C$

Ⓑ $x^2 \ln(x+1) - \dfrac{2x^3}{3(x+1)^2} + C$

Ⓒ $\dfrac{2x}{x+1} - 2\ln(x+1) + C$

Ⓓ $x^2 \ln(x+1) - \dfrac{x^3}{3} \ln(x+1) + C$

Ⓔ $(x^2 - 1)\ln(x+1) - \dfrac{x^2}{2} + x + C$

8. *[3 marks]*

The dollar value 20 years from now of a continuous annuity with payment at time t at the rate of \$500/yr and interest at 5% compounded continuously is

Ⓐ $10000(1 - e)$

Ⓑ $10000\left(1 - \dfrac{1}{e}\right)$

Ⓒ $25(e - 1)$

Ⓓ $500(e - 1)$

Ⓔ $10000(e - 1)$

9. *[3 marks]*
$$\int_2^\infty \frac{1}{x^5}\,dx$$

Ⓐ diverges

Ⓑ equals $\frac{1}{64}$

Ⓒ equals $\frac{16}{3}$

Ⓓ equals $\frac{1}{384}$

Ⓔ equals $-\frac{16}{3}$

10. *[3 marks]*

Approximating $\int_1^4 \sqrt{\ln x}\,dx$ by using the Trapezoidal Rule with $n = 3$, the answer is closest to

Ⓐ 3.058

Ⓑ 1.881

Ⓒ 2.469

Ⓓ 2.393

Ⓔ 1.494

11. *[3 marks]*

Suppose $f(x)$ is a continuous function for all x, and that $\int_4^9 f(x)\,dx = 7$,
then $\int_2^3 xf(x^2)\,dx =$

Ⓐ $\frac{7}{2}$

Ⓑ $\frac{45}{7}$

Ⓒ $\frac{49}{10}$

Ⓓ 7

Ⓔ 9

12. *[3 marks]*

If $\dfrac{dy}{dx} = xy$, and $y = 5$ when $x = 0$, then when $x = 1$, y is closest to

Ⓐ 8.24

Ⓑ 5.65

Ⓒ 5.10

Ⓓ 3.03

Ⓔ 1

13. *[3 marks]*

If $u(x, y) = e^{\frac{x}{y}}$, $x = 2r - s$, $y = r + 2s$ for $r + 2s \neq 0$, then in terms of x and y $\dfrac{\partial u}{\partial r} =$

Ⓐ $\left(\dfrac{2 - x}{y}\right) e^{\frac{x}{y}}$

Ⓑ $\left(\dfrac{2y - x}{y}\right) e^{\frac{x}{y}}$

Ⓒ $3e^{\frac{x}{y}}$

Ⓓ $\dfrac{2}{y} e^{\frac{x}{y}}$

Ⓔ $\left(\dfrac{2y - x}{y^2}\right) e^{\frac{x}{y}}$

14. *[3 marks]*

Given $f(x, y, z) = e^x + x \ln y + y \ln z$. Which is identically equal to zero?

Ⓐ $\dfrac{\partial^2 f}{\partial x^2}$

Ⓑ $\dfrac{\partial^2 f}{\partial x \partial y}$

Ⓒ $\dfrac{\partial^2 f}{\partial z^2}$

Ⓓ $\dfrac{\partial^2 f}{\partial y^2}$

Ⓔ $\dfrac{\partial^2 f}{\partial x \partial z}$

15. *[3 marks]*

The number of critical points of $f(x,y) = 3x^4 - 6xy^2 - 4y^3$ is

Ⓐ 1

Ⓑ 2

Ⓒ 3

Ⓓ 4

Ⓔ 5

PART B. WRITTEN-ANSWER QUESTIONS

B1. *[11 marks]*

Mr. Dollar wishes to accumulate \$20,000 in 16 years by making equal deposits at the end of each year for the 16 years, in an account earning 7% compounded annually.

[4] (a) To the nearest cent, what should the amount of each deposit be?

[7] (b) Just after Mr. Dollar's 8th deposit, the interest rate changes to 6% compounded annually. What should the amount of each of the last 8 equal, annual deposits now be, to attain the original goal of \$20,000?

B2. *[12 marks]*

Find the following:

[6] (a) $\displaystyle\int_0^1 \frac{1}{1+e^x}\,dx$

[6] (b) the area between the graphs of $f(x) = x^3$ and $g(x) = x$ on the interval $[0, 2]$

B3. *[10 marks]*

Evaluate the following integral

$$\int_2^\infty \frac{4x}{(x-1)^2(x+1)}\,dx\ .$$

B4. *[11 marks]*

A candy company produces to types of candy, A and B, for which the costs of production per kg are \$3 and \$5, respectively.

The quantities q_A and q_B (in kg) of A and B that can be sold each week are given by the joint demand functions

$$q_A = 600(p_B - 2p_A)$$
$$q_B = 600(12 + p_A - 2p_B)$$

where p_A and p_B are the selling prices (in dollars per kg) of A and B respectively. Determine the selling prices that will maximize the company's profit. [You only have to show that your answer is a relative max for full marks.]

B5. *[11 marks]*

The production function for a certain manufacturer is

$$z = 100x^{4/5}y^{1/5}$$

where x is the number of units of labour and y is the number of units of capital. Suppose that labour costs \$160 per unit and that capital costs \$200 per unit.

Use the method of Lagrange multipliers to find the number of units of labour and capital that will maximize production, if total expenditure on labour and capital must be \$100,000.

Solutions to April 2001 Exam, MAT133Y

PART A

1. ANSWER: Ⓓ

$$5100 = 5000(1+r)^{\frac{125}{365}}$$

$$\Big[\text{since } (1+i)^{365} = 1+r$$

$$(i+i)^{125} = (1+r)^{\frac{125}{365}}\Big]$$

$$\left(\frac{51}{50}\right)^{\frac{365}{125}} - 1 = r$$

$r \approx .0595281$

$$\boxed{r = 5.95\%}$$

2. ANSWER: Ⓓ

$$\left(\begin{array}{ccc|ccc} 1 & \textcircled{0} & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ -1 & 1 & 0 & 0 & 0 & 1 \end{array}\right) \longrightarrow \left(\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 1 & 0 & 1 \end{array}\right)$$

$$\longrightarrow \left(\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 & -1 & 1 \end{array}\right) \longrightarrow \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 0 & \textcircled{1} & -1 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 & -1 & 1 \end{array}\right)$$

$$0 - 1 = \boxed{-1}$$

3. ANSWER: Ⓒ

$$\left(\begin{array}{ccc|c} 1 & 2 & -3 & 4 \\ 3 & -1 & 5 & 2 \\ 4 & 1 & k^2-14 & k+2 \end{array}\right) \longrightarrow \left(\begin{array}{ccc|c} 1 & 2 & -3 & 4 \\ 0 & -7 & 14 & -10 \\ 0 & 7 & k^2-2 & k-14 \end{array}\right)$$

$$\longrightarrow \left(\begin{array}{ccc|c} 1 & 2 & -3 & 4 \\ 0 & -7 & 14 & -10 \\ 0 & 0 & k^2-16 & k-4 \end{array}\right)$$

If $k \neq \pm 4$, unique solution.
If $k = 4$, ∞-many solutions.
if $k = -4$, no solution.

4. ANSWER: Ⓑ

$$\text{At} \quad x = -1,$$
$$\lim_{x\to -1^-} f(x) = \lim_{x\to -1^-} 2^{\frac{1}{x+1}} + a$$
$$= a$$
$$\lim_{x\to -1^+} f(x) = \lim_{x\to -1^+} \frac{x^2-1}{x-1} = \frac{0}{-2} = 0$$
$$\text{so} \quad a = 0$$
$$\lim_{x\to 1^-} f(x) = \lim_{x\to 1^-} \frac{x^2-1}{x-1} = 2$$
$$\lim_{x\to 1^+} f(x) = \lim_{x\to 1^+} b - x = b - 1$$
$$\text{so} \quad b - 1 = 2 \quad \text{and} \quad b = 3$$

5. ANSWER: Ⓓ

$\ln 1 = 0$ and $e^0 - 1 = 0$ so

L'Hôpital says

$$\lim_{x\to 0} \frac{\ln(1-x)}{e^x - 1} = \lim_{x\to 0} \frac{-\frac{1}{1-x}}{e^x} = -1$$

6. ANSWER: Ⓔ

$$\bar{f} = \frac{1}{10}\int_0^{10} x^9\,dx$$
$$= \frac{1}{10}\cdot \frac{x^{10}}{10}\Big|_0^{10} = \frac{10^{10}}{10^2} = 10^8$$

7. ANSWER: Ⓔ

$$\text{Let} \quad u = \ln(x+1) \quad dv = 2x dx$$
$$du = \frac{1}{x+1}dx \quad v = x^2$$
$$\int = x^2\ln(x+1) - \int \frac{x^2}{x+1}\,dx$$
$$= x^2\ln(x+1) - \int \left[(x-1) - \frac{1}{x+1}\right]dx$$
$$= x^2\ln(x+1) - \frac{x^2}{2} + x - \ln(x+1) + C$$
$$= (x^2-1)\ln(x+1) - \frac{x^2}{2} + x + C$$

8. ANSWER: Ⓔ

$$\begin{aligned} &\int_0^{20} 500e^{.05(20-t)}\,dt \\ =&\int_0^{20} 500e^{1-.05t}\,dt \\ =&500\frac{e^{1-.05t}}{-.05}\Big|_0^{20} \\ =&\frac{500}{.05}(e-e^0) \\ =&10,000(e-1) \end{aligned}$$

9. ANSWER: Ⓑ

$$\begin{aligned} &= \lim_{R\to\infty} \int_2^R x^{-5}dx \\ &= \lim_{R\to\infty} \frac{x^{-4}}{-4}\Big|_2^R \\ &= \lim_{R\to\infty} \left(-\frac{1}{4R^4}+\frac{1}{4\cdot 16}\right) \\ &= \frac{1}{64} \end{aligned}$$

10. ANSWER: Ⓒ

$$\begin{aligned} T_3 &= \frac{4-1}{6}\big[f(1)+2f(2)+2f(3)+f(4)\big] \\ &= \frac{1}{2}\big[\sqrt{\ln 1}+2\sqrt{\ln 2}+2\sqrt{\ln 3}+\sqrt{\ln 4}\big] \\ &\approx 2.4694 \end{aligned}$$

11. ANSWER: Ⓐ

$$\text{Let} \quad u = x^2 \qquad \frac{du}{2} = x\,dx$$

$$\int_2^3 xf(x^2)dx = \frac{1}{2}\int_4^9 f(u)du = \frac{7}{2}$$

12. ANSWER: Ⓐ

$$\begin{aligned} \frac{dy}{y} &= x\,dx \\ \ln y &= \frac{x^2}{2}+C \\ y &= Ae^{x^2/2} \\ 5 &= Ae^0 \\ y &= 5e^{x^2/2} \\ y(1) &= 5e^{1/2} \approx 8.2436 \end{aligned}$$

13. ANSWER: Ⓔ

$$
\begin{aligned}
\frac{\partial u}{\partial r} &= \frac{\partial u}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial r} \\
&= \frac{1}{y} e^{\frac{x}{y}} \cdot 2 + \left(-\frac{x}{y^2}\right) e^{\frac{x}{y}} \cdot 1 \\
&= e^{\frac{x}{y}} \left(-\frac{x}{y^2} + \frac{2}{y}\right) \\
&= \left(\frac{2y - x}{y^2}\right) e^{\frac{x}{y}}
\end{aligned}
$$

14. ANSWER: Ⓔ

$$\frac{\partial f}{\partial x} = e^x + \ln y$$

so $\dfrac{\partial^2 f}{\partial x \partial z} = \dfrac{\partial^2 f}{\partial z \partial x} = 0$

None of the others are.

15. ANSWER: Ⓑ

$f_x = 12x^3 - 6y^2 = 6(2x^3 - y^2)$

$f_y = -12xy - 12y^2 = -12y(x + y)$

$f_y = 0 \quad \Rightarrow \quad y = 0 \quad \text{or} \quad y = -x$

If $\quad y = 0,\ f_x = 0 \quad \Rightarrow \quad x = 0 \qquad (0, 0)$ crit.

If $\quad y = -x,\ f_x = 0 \quad \Rightarrow$

$2x^3 - x^2 = 0$

$x^2(2x - 1) = 0 \quad \Rightarrow \quad x = 0,\ y = 0 \quad$ (nothing new)

or $\quad x = \dfrac{1}{2}$

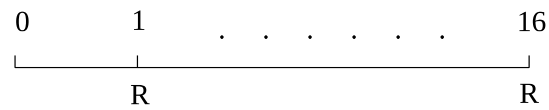
$y = -\dfrac{1}{2} \qquad \left(\dfrac{1}{2}, -\dfrac{1}{2}\right)$

2 critical points

PART B

B1.

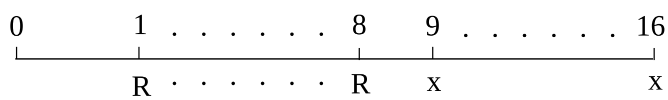
(a)



$$R = s_{\overline{16}|.07} = 20,000$$

$$R = \frac{20,000 \times .07}{(1.07)^{16} - 1} = \boxed{\$\ 717.15}$$

(b)



$$R = s_{\overline{8}|.07}(1.06)^8 + X s_{\overline{8}|.06} = 20,000$$

$$X = \frac{20,000 - 717.15 s_{\overline{8}|.07}(1.06)^8}{s_{\overline{8}|.06}}$$

$$\boxed{\text{X} = \$\ 835.85}$$

B2.

(a) $\displaystyle\int_0^1 \frac{1}{1+e^x}\,dx$

$$= \left[x - \ln(1+e^x)\right]_0^1$$

$$= \boxed{1 - \ln(1+e) + \ln 2}$$

$$= \approx .38$$

$u = 1 + e^x \qquad du = e^x dx = (u-1)dx$

$$\frac{du}{u-1} = dx$$

$$\int \frac{1}{1+e^x} dx = \int \frac{du}{u(u-1)}$$

$$= \int \left(\frac{1}{u-1} - \frac{1}{u}\right) du = \ln\left(\frac{u-1}{u}\right)$$

$$= \ln\left(\frac{e^x}{1+e^x}\right) = x - \ln(1+e^x)$$

OR:

$$\int_0^1 \frac{1}{1+e^x}dx = \int_0^1 \frac{e^{-x}}{e^{-x}+1}dx = -\ln(1+e^{-x})\Big|_0^1$$

$$= \boxed{-\ln\left(1+\tfrac{1}{e}\right)+\ln 2}$$

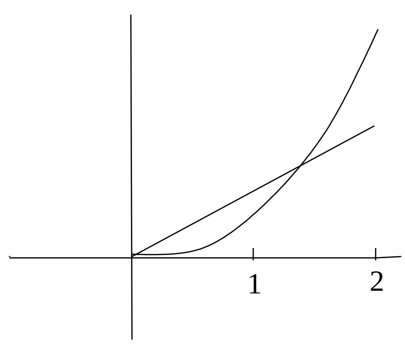
$$= -\ln\left(\frac{e+1}{e}\right)+\ln 2$$

$$= -\ln(e+1)+\ln e+\ln 2$$

$$= -\ln(e+1)+1+\ln 2$$

the same

(b)



$$A = \int_0^1 (x-x^3)dx + \int_1^2 (x^3-x)\,dx$$

$$= \left(\frac{1}{2}-\frac{1}{4}\right)+\frac{2^4-1}{4}-\frac{2^2-1}{2}$$

$$= \boxed{\frac{5}{2}}$$

B3.

$$\frac{4x}{(x-1)^2(x+1)} = \frac{A}{x-1}+\frac{B}{(x-1)^2}+\frac{C}{x+1}$$

$$A(x-1)(x+1)+B(x+1)+C(x-1)^2 = 4x$$

$x = 1 \Rightarrow 2B = 4 \Rightarrow B = 2$

$x = -1 \Rightarrow 4C = -4 \Rightarrow C = -1$

$x = 0 \Rightarrow -A+B+C = 0 \Rightarrow A = B+C = 1$

$$\int_2^\infty \frac{4x}{(x-1)^2(x+1)}dx = \lim_{R\to\infty}\int_2^R \left(\frac{1}{x-1}+\frac{2}{(x-1)^2}-\frac{1}{x+1}\right)dx$$

$$= \lim_{R\to\infty}\left[\ln\left|\frac{x-1}{x+1}\right|-\frac{2}{x-1}\right]_2^R$$

$$= \lim_{R\to\infty}\left[\ln\left|\frac{R-1}{R+1}\right|-\frac{2}{R-1}-\left(\ln\frac{1}{3}-2\right)\right]$$

but $\dfrac{R-1}{R+1}\to 1$ as $R\to\infty$ and $\dfrac{2}{R-1}\to 0$ as $R\to\infty$ so get $\boxed{2+\ln 3}$

B4.

$$\pi = p_Aq_A + p_Bq_B - C(q_A, q_B)$$

$$C(q_A, q_B) = 3q_A + 5q_B$$

$$\begin{aligned}
\pi &= p_Aq_A + p_Bq_B - 3q_A - 5q_B \\
&= (p_A - 3)q_A + (p_B - 5)q_B \\
&= 600(p_B - 2p_A)(p_A - 3) + 600(12 + p_A - 2p_B)(p_B - 5) \\
\frac{\partial \pi}{\partial p_A} &= 600\big[\{-2(p_A - 3) + p_B - 2p_A\} + \{(p_B - 5)\}\big] \\
&= 600\big[-4p_A + 2p_B + 1\big] \\
\frac{\partial \pi}{\partial p_B} &= 600\big[\{(p_A - 3)\} + \{-2(p_B - 5) + (12 + p_A - 2p_B)\}\big] \\
&= 600\big[2p_A - 4p_B + 19\big]
\end{aligned}$$

$$\left.\begin{aligned} 4p_A - 2p_B &= 1 \\ 2p_A - 4p_B &= -19 \end{aligned}\right\} \quad \begin{aligned} 6p_B &= 39 \\ p_B &= 6.5 \\ -6p_A &= -21 \\ p_A &= 3.5 \end{aligned}$$

$$\frac{\partial^2 \pi}{\partial p_A^2} = -4 \times 600 \qquad \frac{\partial^2 \pi}{\partial p_B^2} = -4 \times 600 \qquad \frac{\partial^2 \pi}{\partial p_A \partial p_B} = 2 \times 600$$

$D = (-4)^2 \times 600^2 - 2^2(600)^2 = 12 \times 600^2 > 0$ and $\dfrac{\partial^2 \pi}{\partial p_A^2} < 0$ so local max at

$$\boxed{p_A = \$3.50/\text{kg} \;\text{ and }\; p_B = \$6.50/\text{kg}}$$

B5.

$$\mathcal{L} = 100x^{4/5}y^{1/5} - \lambda(160x + 200y - 100,000)$$

$$\mathcal{L}_x = 80x^{-1/5}y^{1/5} - 160\lambda = 0$$

$$\mathcal{L}_y = 20x^{4/5}y^{-4/5} - 200\lambda = 0$$

$$160x + 200y = 100,000$$

$$\left(\frac{y}{x}\right)^{1/5} = 2\lambda$$

$$\left(\frac{x}{y}\right)^{4/5} = 10\lambda$$

and since $\lambda = 0$ forces x or y to be zero which gives zero production, $\lambda \neq 0$.

So $$\left(\frac{x}{y}\right)^{1/5} = \frac{1}{2\lambda}$$

$$\left(\frac{x}{y}\right)^{1/5}\left(\frac{x}{y}\right)^{4/5} = 5$$

$$x = 5y$$

$$800y + 200y = 10,000$$

$$\boxed{\begin{aligned} y &= 100 \\ x &= 500 \end{aligned}}$$