Solutions to Supplementary Questions for HP Chapter 6

1. We have

$$\begin{array}{lll} ① & a-b & =8 \\ ② & b+c & =1 \\ ③ & 3d+c & =7 \\ ④ & 2a-4d & =6 \end{array}$$

Adding ① and ②, we get: ⑤ $a+c=9$. By taking ⑤ − ③, we get ⑥ $a-3d=2$. Now, ④ − [2 × ⑥] gives us ⑦ $2d=2$, or $d=1$. Substituting back into ④, we get $a=5$. Similarly, $c=4$ and $b=-3$. So, $a=5$, $b=-3$, $c=4$, $d=1$.

2. (a) $$Q_{d1}\begin{bmatrix}1\\1\\0\\0\\0\\0\end{bmatrix}+Q_{s1}\begin{bmatrix}-1\\0\\1\\0\\0\\0\end{bmatrix}+Q_{d2}\begin{bmatrix}0\\0\\0\\1\\1\\0\end{bmatrix}+Q_{s2}\begin{bmatrix}0\\0\\0\\-1\\0\\1\end{bmatrix}+P_1\begin{bmatrix}0\\-a_1\\-b_1\\0\\-\alpha_1\\-\beta_1\end{bmatrix}+P_2\begin{bmatrix}0\\-a_2\\-b_2\\0\\-\alpha_2\\-\beta_2\end{bmatrix}=\begin{bmatrix}0\\a_0\\b_0\\0\\\alpha_0\\\beta_0\end{bmatrix}.$$

(b) $$\begin{bmatrix}1&-1&0&0&0&0\\1&0&0&0&-a_1&-a_2\\0&1&0&0&-b_1&-b_2\\0&0&1&-1&0&0\\0&0&1&0&-\alpha_1&-\alpha_2\\0&0&0&1&-\beta_1&-\beta_2\end{bmatrix}\begin{bmatrix}Q_{d1}\\Q_{s1}\\Q_{d2}\\Q_{s2}\\P_1\\P_2\end{bmatrix}=\begin{bmatrix}0\\a_0\\b_0\\0\\\alpha_0\\\beta_0\end{bmatrix}$$

3. $A^2 = \begin{pmatrix} 6 & 9 & 0 \\ -4 & -6 & 0 \\ 1 & 3 & 1 \end{pmatrix} \begin{pmatrix} 6 & 9 & 0 \\ -4 & -6 & 0 \\ 1 & 3 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ -5 & -6 & 1 \end{pmatrix}$

$$\begin{aligned}
A^4 &= A^2 \cdot A^2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ -5 & -6 & 1 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ -5 & -6 & 1 \end{pmatrix} \\
&= \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ -5 & -6 & 1 \end{pmatrix} = A^2 \\
A^8 &= A^4 \cdot A^4 = A^2 \cdot A^2 = A^4 = A^2 \\
A^{16} &= A^8 \cdot A^8 = A^2 \cdot A^2 = A^4 = A^2 \\
A^{32} &= A^{16} \cdot A^{16} = A^2 \cdot A^2 = A^4 = A^2 \\
A^{64} &= A^{32} \cdot A^{32} = A^2 \cdot A^2 = A^4 = A^2 \\
A^{65} &= A \cdot A^{64} = A \cdot A^2 = \begin{pmatrix} 6 & 9 & 0 \\ -4 & -6 & 0 \\ 1 & 3 & 1 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ -5 & -6 & 1 \end{pmatrix} \\
&= \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ -5 & -6 & 1 \end{pmatrix}
\end{aligned}$$

Another method:

From just above, we have seen that $A^3 = A \cdot A^2 = A^2$. But now, $A^4 = A \cdot (A \cdot A^2) = A \cdot A^2$, and in fact for any $n \geq 2$,

$$\begin{aligned}
A^n &= \overbrace{A \cdot (A \cdot (A \cdot (\ldots (A \cdot}^{n-3 \text{ times}}(A \cdot A^2)) \ldots) \\
&= \overbrace{A \cdot (A \cdot (A \cdot (\ldots}^{n-4 \text{ times}}(A \cdot (A^2)) \ldots) \\
&= \overbrace{A \cdot (A \cdot (A \cdot (\ldots}^{n-5 \text{ times}}(A \cdot A^2) \ldots) = \ldots = \\
&= A \cdot (A \cdot (A \cdot A^2)) = A \cdot (A \cdot A^2) = A \cdot A^2 = A^2.
\end{aligned}$$

In particular, when $n = 65$, $A^{65} = A^2$.

4. (a) The matrix forms are

$$Y = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 1 & -1 & 1 \\ 3 & 1 & -4 \\ -2 & -2 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = AX,$$

$$Z = \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} = \begin{bmatrix} 4 & -1 & 1 \\ -3 & 5 & -1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = BY$$

$Z = BY = BAX$, so

$$C = BA = \begin{bmatrix} 4 & -1 & 1 \\ -3 & 5 & -1 \end{bmatrix} \begin{bmatrix} 1 & -1 & 1 \\ 3 & 1 & -4 \\ -2 & -2 & 3 \end{bmatrix} = \begin{bmatrix} -1 & -7 & 11 \\ 14 & 10 & -26 \end{bmatrix}$$

and

$$Z = \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} = \begin{bmatrix} -1 & -7 & 11 \\ 14 & 10 & -26 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = CX$$

(b) From the above equation,

$$\begin{array}{rcrrr} z_1 & = & -x_1 & -7x_2 & +11x_3 \\ z_2 & = & 14x_1 & +10x_2 & -26x_3 \end{array}$$

(c) Making the substitutions,

$$\begin{aligned} z_1 &= 4y_1 - y_2 + y_3 \\ &= 4(x_1 - x_2 + x_3) - (3x_1 + x_2 - 4x_3) + (-2x_1 - 2x_2 + 3x_3) \\ &= 4x_1 - 4x_2 + 4x_3 - 3x_1 - x_2 + 4x_3 - 2x_1 - 2x_2 + 3x_3 \\ &= -x_1 - 7x_2 + 11x_3 \\ z_2 &= -3y_1 + 5y_2 - y_3 \\ &= -3(x_1 - x_2 + x_3) + 5(3x_1 + x_2 - 4x_3) - (-2x_1 - 2x_2 + 3x_3) \\ &= -3x_1 + 3x_2 - 3x_3 + 15x_1 + 5x_2 - 20x_3 + 2x_1 + 2x_2 - 3x_3 \\ &= 14x_1 + 10x_2 - 26x_3 \end{aligned}$$

Both z_1 and z_2 agree with the results in (b).

5.

(a) $AX = kX \Rightarrow \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = k \begin{bmatrix} x \\ y \end{bmatrix} \Rightarrow \begin{bmatrix} y \\ x \end{bmatrix} = \begin{bmatrix} kx \\ ky \end{bmatrix} \Rightarrow$ ① $y = kx$ and ② $x = ky$.
By substituting ② into ①, we get $y = k(ky) = k^2 y$. Similarly, we see that $x = k^2 x$.
Since either x or y is non-zero, this forces $k^2 = 1$, and hence $k = 1$ or $k = -1$.

(b) i) $k = 1$. Here $AX = kX \Rightarrow AX = X \Rightarrow \begin{bmatrix} y \\ x \end{bmatrix} = \begin{bmatrix} x \\ y \end{bmatrix}$. In other words, $x = y$.

So all X of the form $X = \begin{bmatrix} x \\ x \end{bmatrix}$, for any $x \neq 0$ satisfies $AX = kX$ when $k = 1$.

ii) $k = -1$. Here $AX = kX \Rightarrow AX = -X \Rightarrow \begin{bmatrix} y \\ x \end{bmatrix} = \begin{bmatrix} -x \\ -y \end{bmatrix}$. In other words, $y = -x$.

So all X of the form $X = \begin{bmatrix} x \\ -x \end{bmatrix}$, for any $x \neq 0$ satisfies $AX = kX$ when $k = -1$.

6.

$$\begin{array}{llll} x & +y & +z & = 20 \\ 4x & +6y & +8z & = 108 \\ \frac{1}{2}(4x) & +\frac{1}{2}(6y) & +\frac{1}{4}(8z) & = 46 \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 20 \\ 4 & 6 & 8 & 108 \\ 2 & 3 & 2 & 46 \end{array}\right]$$

$$\longrightarrow \left[\begin{array}{ccc|c} 1 & 1 & 1 & 20 \\ 0 & 2 & 4 & 28 \\ 0 & 1 & 0 & 6 \end{array}\right] \begin{array}{l} \\ -4R_1 + R_2 \\ -2R_1 + R_3 \end{array}$$

$$\longrightarrow \left[\begin{array}{ccc|c} 1 & 1 & 1 & 20 \\ 0 & 1 & 0 & 6 \\ 0 & 2 & 4 & 28 \end{array}\right] \quad R_2 \leftrightarrow R_3$$

$$\longrightarrow \left[\begin{array}{ccc|c} 1 & 0 & 1 & 14 \\ 0 & 1 & 0 & 6 \\ 0 & 0 & 4 & 16 \end{array}\right] \begin{array}{l} R_1 - R_2 \\ \\ -2R_2 + R_3 \end{array}$$

$$\longrightarrow \left[\begin{array}{ccc|c} 1 & 0 & 1 & 14 \\ 0 & 1 & 0 & 6 \\ 0 & 0 & 1 & 4 \end{array}\right] \begin{array}{l} \\ \\ \frac{1}{4}R_3 \end{array}$$

$$\longrightarrow \left[\begin{array}{ccc|c} 1 & 0 & 0 & 10 \\ 0 & 1 & 0 & 6 \\ 0 & 0 & 1 & 4 \end{array}\right] \begin{array}{l} R_1 - R_3 \\ \\ \\ \end{array}$$

Therefore $x = 10$, $y = 6$, $z = 4$.

7. (a) The second equation is the negative of the first, so there are really only two distinct equations

$$(q-1)x + py = 0$$
$$x + y = 1$$

which yield the augmented coefficient matrix

$$\left[\begin{array}{cc|c} q-1 & p & 0 \\ 1 & 1 & 1 \end{array}\right]$$

Adding $1 - q$ times the second row to the first and interchanging the two rows gives the matrix

$$\left[\begin{array}{cc|c} 1 & 1 & 1 \\ 0 & 1-q+p & 1-q \end{array}\right] \tag{$*$}$$

Since $p > 0$, $q < 1$, then $1 - q + p \neq 0$ so we may add $-\frac{1}{1-q+p}$ times the second row to the first

$$\left[\begin{array}{cc|c} 1 & 0 & \frac{q-1}{1-q+p} + 1 \\ 0 & 1-q+p & 1-q \end{array}\right]$$

and multiplying the second row by $\frac{1}{1-q+p}$ gives

$$\left[\begin{array}{cc|c} 1 & 0 & \frac{q-1}{1-q+p}+1 \\ 0 & 1 & \frac{1-q}{1-q+p} \end{array}\right]$$

or

$$\left[\begin{array}{cc|c} 1 & 0 & \frac{p}{1-q+p} \\ 0 & 1 & \frac{1-q}{1-q+p} \end{array}\right]$$

so there is the unique solution $x = \frac{p}{1-q+p}$, $y = \frac{1-q}{1-q+p}$.

(b) The above argument is valid unless $1 - q + p = 0$, or $p - q = -1$. If that is the case, the augmented matrix $(*)$ becomes

$$\left[\begin{array}{cc|c} 1 & 1 & 1 \\ 0 & 0 & 1-q \end{array}\right]$$

If $q \neq 1$, this system has no solutions. If $q = 1$ (and hence $p = 0$), then the matrix is

$$\left[\begin{array}{cc|c} 1 & 1 & 1 \\ 0 & 0 & 0 \end{array}\right]$$

and any x, y such that $x + y = 1$ is a solution. Therefore the system has no solutions for any p, q such that $p - q = -1$ and $q \neq 1$.

8. Let x be the number of pennies, y the number of nickels, and z the number of dimes. Since there are 13 coins, $x+y+z = 13$, and since their value is 83 cents, $x+5y+10z = 83$.

Therefore the problem is to find *non-negative, integer* solutions to the system

$$\begin{array}{cccc} x & +y & +z & = 13 \\ x & +5y & +10z & = 83 \end{array}.$$

The augmented coefficient matrix is

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 13 \\ 1 & 5 & 10 & 83 \end{array}\right]$$

which we now reduce

$$\begin{aligned} &\left[\begin{array}{ccc|c} 1 & 1 & 1 & 13 \\ 1 & 5 & 10 & 83 \end{array}\right] \\ \longrightarrow &\left[\begin{array}{ccc|c} 1 & 1 & 1 & 13 \\ 0 & 4 & 9 & 70 \end{array}\right] -R_1 + R_2 \\ \longrightarrow &\left[\begin{array}{ccc|c} 1 & 1 & 1 & 13 \\ 0 & 1 & \frac{9}{4} & \frac{70}{4} \end{array}\right] \tfrac{1}{4}R_2 \\ \longrightarrow &\left[\begin{array}{ccc|c} 1 & 0 & -\frac{5}{4} & -\frac{18}{4} \\ 0 & 1 & \frac{9}{4} & \frac{70}{4} \end{array}\right] -R_2 + R_1 \end{aligned}$$

The solution in parametric form is $z = r$, $y = \frac{70-9r}{4}$, $x = \frac{-18+5r}{4}$. We now try non-negative integer values of r to determine which value(s) of r give non-negative integer values for x and y. For $r \leq 3$, x is negative, and for $r \geq 8$ y is negative, so we need only try $r = 4, 5, 6, 7$. A quick check shows $r = 6$ gives the only integer solution, which is $x = 3$, $y = 4$, $z = 6$.

Therefore, there are three pennies, four nickels and six dimes in the box.

9. The augmented coefficient matrix of the system is

$$\left[\begin{array}{cccccc|c} 1 & 3 & -2 & 0 & 2 & 0 & 0 \\ 2 & 6 & -5 & -2 & 4 & -3 & -1 \\ 0 & 0 & 5 & 10 & 0 & 15 & 5 \\ 2 & 6 & 0 & 8 & 4 & 18 & 6 \end{array}\right]$$

$$\longrightarrow \left[\begin{array}{cccccc|c} 1 & 3 & -2 & 0 & 2 & 0 & 0 \\ 0 & 0 & -1 & -2 & 0 & -3 & -1 \\ 0 & 0 & 5 & 10 & 0 & 15 & 5 \\ 0 & 0 & 4 & 8 & 0 & 18 & 6 \end{array}\right] \begin{array}{l} \\ -2R_1 + R_2 \\ \\ -2R_1 + R_4 \end{array}$$

$$\longrightarrow \left[\begin{array}{cccccc|c} 1 & 3 & -2 & 0 & 2 & 0 & 0 \\ 0 & 0 & 1 & 2 & 0 & 3 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 6 & 2 \end{array}\right] \begin{array}{l} \\ \\ 5R_2 + R_3 \\ 4R_2 + R_4 \end{array} \quad \textbf{then} - R_2$$

$$\longrightarrow \left[\begin{array}{cccccc|c} 1 & 3 & -2 & 0 & 2 & 0 & 0 \\ 0 & 0 & 1 & 2 & 0 & 3 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & \frac{1}{3} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right] \quad R_3 \leftrightarrow R_4 \quad \textbf{then} \tfrac{1}{6} R_3$$

(Note : this matrix is said to be in 'reduced row echelon form')

$$\longrightarrow \left[\begin{array}{cccccc|c} 1 & 3 & -2 & 0 & 2 & 0 & 0 \\ 0 & 0 & 1 & 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & \frac{1}{3} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right] \begin{array}{l} \\ -3R_3 + R_2 \\ \\ \\ \end{array}$$

$$\longrightarrow \left[\begin{array}{cccccc|c} 1 & 3 & 0 & 4 & 2 & 0 & 0 \\ 0 & 0 & 1 & 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & \frac{1}{3} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right] \begin{array}{l} 2R_2 + R_1 \\ \\ \\ \\ \end{array}$$

Discarding the last unnecessary row of zeros, the corresponding system of equations is

$$\begin{aligned} x_1 + 3x_2 + 4x_4 + 2x_5 &= 0 \\ x_3 + 2x_4 &= 0 \\ x_6 &= \frac{1}{3} \end{aligned}$$

from which,

$$x_1 = -3x_2 - 4x_4 - 2x_5$$
$$x_3 = -2x_4$$
$$x_6 = \frac{1}{3}$$

There is no restriction placed on x_2, x_4, or x_5, so we get as a solution set

$$x_1 = -3r - 4s - 2t$$
$$x_2 = r$$
$$x_3 = -2s$$
$$x_4 = s$$
$$x_5 = t$$
$$x_6 = \frac{1}{3}$$

where r, s and t are any real numbers.

10. Since this is a homogeneous system, we find the reduced coefficient matrix of this system

$$\begin{bmatrix} 1 & 3 & 2 \\ 1 & c & 4 \\ 0 & 2 & c \end{bmatrix}$$
$$\longrightarrow \begin{bmatrix} 1 & 3 & 2 \\ 0 & c-3 & 2 \\ 0 & 2 & c \end{bmatrix} \; -R_1 + R_2$$
$$\longrightarrow \begin{bmatrix} 1 & 3 & 2 \\ 0 & 2 & c \\ 0 & c-3 & 2 \end{bmatrix} \; R_2 \leftrightarrow R_3$$
$$\longrightarrow \begin{bmatrix} 1 & 0 & \frac{4-3c}{2} \\ 0 & 1 & \frac{c}{2} \\ 0 & 0 & \frac{4-c(c-3)}{2} \end{bmatrix} \; \begin{matrix} -\frac{3}{2}R_2 + R_1 \\ \textbf{Also}, \frac{1}{2}R_2 \\ -\frac{c-3}{2}R_2 + R_3 \end{matrix} \quad \textcircled{A}$$

At this point, we note that if the last row is non-zero (i.e., if $\frac{4-c(c-3)}{2} \neq 0$) then we can continue as follows

$$\longrightarrow \begin{bmatrix} 1 & 0 & \frac{4-3c}{2} \\ 0 & 1 & \frac{c}{2} \\ 0 & 0 & 1 \end{bmatrix} \; \frac{2}{4-c(c-3)}R_3$$
$$\longrightarrow \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \; \begin{matrix} -\frac{(4-3c)}{2}R_3 + R_1 \\ -\frac{c}{2}R_3 + R_2 \end{matrix}$$

So if $\frac{4-c(c-3)}{2} \neq 0$ then the number of non-zero rows is equal to the number of unknowns in this reduced matrix and hence the trivial solution is the only solution. However, if at Ⓐ $\frac{4-c(c-3)}{2} = 0$, i.e., $c^2 - 3c - 4 = 0$, or $(c-4)(c+1) = 0$, whence $c = 4$ or $c = -1$, then the matrix at Ⓐ looks as follows

$$\begin{bmatrix} 1 & 0 & \frac{4-3c}{2} \\ 0 & 1 & \frac{c}{2} \\ 0 & 0 & 0 \end{bmatrix}$$

and this is a reduced matrix with fewer non-zero rows than unknowns.

Thus, if $c = -1$ or $c = 4$, the system has infinitely many solutions. And if $c \neq -1, 4$, then the system has only the trivial solution.

11. (a) Here, the associated homogeneous system is

$$\begin{array}{rrrrl} 2x_1 & +x_2 & -3x_3 & +x_4 & = 0 \quad ① \\ & 5x_2 & +4x_3 & +3x_4 & = 0 \quad ② \\ & & x_3 & +2x_4 & = 0 \quad ③ \\ & & & 3x_4 & = 0 \quad ④ \end{array}$$

From ④ , $x_4 = 0$
From ③ , $x_3 + 2(0) = 0$, hence $x_3 = 0$
From ② , $5x_2 + 4(0) + 3(0) = 0$, hence $x_2 = 0$
From ① , $2x_1 + 0 - 3(0) + 0 = 0$, hence $x_1 = 0$

Hence, there is only one unique (trivial) solution and by the hint, A is invertible.

(b) The associated homogeneous system is

$$\begin{array}{rrrrl} 5x_1 & +x_2 & +4x_3 & +x_4 & = 0 \quad ① \\ & & 2x_3 & -x_4 & = 0 \quad ② \\ & & x_3 & +x_4 & = 0 \quad ③ \\ & & & 7x_4 & = 0 \quad ④ \end{array}$$

From ④ , $x_4 = 0$
From ③ , $x_3 + 0 = 0$, hence $x_3 = 0$
From ② , $0 = 0$

From ① $5x_1 + x_2 + 4(0) + 0 = 0$, hence $5x_1 + x_2 = 0$. In other words, $x_1 = \frac{-x_2}{5}$. Now, for any real r,

$$x_1 = -\frac{r}{5}$$
$$x_2 = r$$
$$x_3 = 0$$
$$x_4 = 0$$

is a solution. But from the text, (HP, p. 281), if B were an invertible matrix, then the above system would have a unique solution, which it does not. Thus B cannot be an invertible matrix.

12. (a) (i) If the inverse of $I - A$ exists, it must be unique, so we need only show that

$$(I - A)(I + A + A^2 + A^3) = I \qquad \text{if} \quad A^4 = 0$$

From the properties of matrix multiplication and addition

$$\begin{aligned}(I - A)(I + A + A^2 + A^3) &= I + A + A^2 + A^3 - A - A^2 - A^3 - A^4 \\ &= I - A^4 \\ &= I \qquad \text{if} \quad A^4 = 0\end{aligned}$$

(ii) As in (i), we have

$$\begin{aligned}&(I - A)(I + A + A^2 + \cdots + A^{n-1}) \\ &\quad = (I + A + A^2 + \cdots + A^{n-1}) - (A + A^2 + \cdots + A^n) \\ &\quad = I + (A - A) + (A^2 - A^2) + \cdots + (A^{n-1} - A^{n-1}) - A^n \\ &\quad = I - A^n \\ &\quad = I \qquad \text{if} \quad A^n = 0\end{aligned}$$

(b) If the inverse of $I - J_n$ exists, it must be unique, so we need to show

$$(I - J_n)\left(I - \left(\frac{1}{n-1}\right) J_n\right) = I$$

We have

$$(I - J_n)\left(I - \frac{1}{n-1} J_n\right) = I - \frac{1}{n-1} J_n - J_n + \frac{1}{n-1} J_n^2$$

Now J_n^2 is the matrix with n in every position, so $J_n^2 = nJ_n$. Therefore,

$$\begin{aligned}(I - J_n)\left(I - \frac{1}{n-1} J_n\right) &= I - \frac{1}{n-1} J_n - J_n + \frac{n}{n-1} J_n \\ &= I + \frac{n - (n-1) - 1}{n-1} J_n \\ &= I + 0J_n \\ &= I\end{aligned}$$

and we're done.

13. From the properties of matrix multiplication and addition:

$$\begin{aligned}
&A^3 - 3A^2 + 2A + I = 0 \\
&\Rightarrow A^3 - 3A^2 + 2A = -I \\
&\Rightarrow A(A^2 - 3A + 2) = -I \\
&\Rightarrow A(A - I)(A - 2I) = -I \\
&\Rightarrow A(I - A)(A - 2I) = I
\end{aligned}$$

But now it is obvious that A is invertible and that $A^{-1} = (I - A)(A - 2I)$. So the answer is (d).

14.(a) The technology matrix is

$$\begin{array}{cc} & \text{Input} \\ & \begin{array}{ccc} A & B & C \end{array} \\ \text{Output} \quad \begin{array}{c} A \\ B \\ C \end{array} & \begin{pmatrix} \frac{1}{4} & \frac{1}{4} & \frac{1}{4} \\ \frac{1}{4} & \frac{1}{4} & \frac{1}{4} \\ \frac{1}{4} & \frac{1}{4} & \frac{1}{4} \end{pmatrix} = A \end{array}$$

Hence, the Leontiff matrix, $I - A$, is

$$\begin{pmatrix} \frac{3}{4} & -\frac{1}{4} & -\frac{1}{4} \\ -\frac{1}{4} & \frac{3}{4} & -\frac{1}{4} \\ -\frac{1}{4} & -\frac{1}{4} & \frac{3}{4} \end{pmatrix}$$

We solve:

$$(I - A) \begin{pmatrix} x_A \\ x_B \\ x_C \end{pmatrix} = \begin{pmatrix} d_A \\ d_B \\ d_C \end{pmatrix}$$

where x_A, x_B, x_C are the productions of industries A, B and C respectively. So

$$\begin{pmatrix} x_A \\ x_B \\ x_C \end{pmatrix} = (I - A)^{-1} \begin{pmatrix} d_A \\ d_B \\ d_C \end{pmatrix}$$

We calculate $(I-A)^{-1}$

$$\left(\begin{array}{ccc|ccc} \frac{3}{4} & -\frac{1}{4} & -\frac{1}{4} & 1 & 0 & 0 \\ -\frac{1}{4} & \frac{3}{4} & -\frac{1}{4} & 0 & 1 & 0 \\ -\frac{1}{4} & -\frac{1}{4} & \frac{3}{4} & 0 & 0 & 1 \end{array}\right)$$

$$\longrightarrow \left(\begin{array}{ccc|ccc} 3 & -1 & -1 & 4 & 0 & 0 \\ -1 & 3 & -1 & 0 & 4 & 0 \\ -1 & -1 & 3 & 0 & 0 & 4 \end{array}\right)$$

$$\longrightarrow \left(\begin{array}{ccc|ccc} 1 & 1 & -3 & 0 & 0 & -4 \\ 3 & -1 & -1 & 4 & 0 & 0 \\ -1 & 3 & -1 & 0 & 4 & 0 \end{array}\right)$$

$$\longrightarrow \left(\begin{array}{ccc|ccc} 1 & 1 & -3 & 0 & 0 & -4 \\ 0 & -4 & 8 & 4 & 0 & 12 \\ 0 & 4 & -4 & 0 & 4 & -4 \end{array}\right)$$

$$\longrightarrow \left(\begin{array}{ccc|ccc} 1 & 1 & -3 & 0 & 0 & -4 \\ 0 & 1 & -2 & -1 & 0 & -3 \\ 0 & 0 & 4 & 4 & 4 & 8 \end{array}\right)$$

$$\longrightarrow \left(\begin{array}{ccc|ccc} 1 & 0 & -1 & 1 & 0 & -1 \\ 0 & 1 & -2 & -1 & 0 & -3 \\ 0 & 0 & 1 & 1 & 1 & 2 \end{array}\right)$$

$$\longrightarrow \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 2 & 1 & 1 \\ 0 & 1 & 0 & 1 & 2 & 1 \\ 0 & 0 & 1 & 1 & 1 & 2 \end{array}\right)$$

$$\Rightarrow (I-A)^{-1} = \begin{pmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{pmatrix}$$

So

$$\begin{pmatrix} x_A \\ x_B \\ x_C \end{pmatrix} = \begin{pmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{pmatrix} \begin{pmatrix} d_A \\ d_B \\ d_C \end{pmatrix} = \begin{pmatrix} 2d_A + d_B + d_C \\ d_A + 2d_B + d_C \\ d_A + d_B + 2d_C \end{pmatrix}$$

(b) Here, the technology matrix is

$$\begin{pmatrix} \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \end{pmatrix} = \frac{1}{3} \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix},$$

so $(I-A)$, the Leontiff matrix, is

$$I - A = \frac{1}{3} \begin{pmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{pmatrix}$$

We solve

$$\frac{1}{3}\begin{pmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{pmatrix}\begin{pmatrix} x_A \\ x_B \\ x_C \end{pmatrix} = \begin{pmatrix} d_A \\ d_B \\ d_C \end{pmatrix}$$

Hence,

$$\begin{aligned}
&\frac{1}{3}\left(\begin{array}{ccc|c} 2 & -1 & -1 & 3d_A \\ -1 & 2 & -1 & 3d_B \\ -1 & -1 & 2 & 3d_C \end{array}\right) \\
&\longrightarrow \frac{1}{3}\left(\begin{array}{ccc|c} 1 & 1 & -2 & -3d_C \\ 0 & 3 & -3 & 3d_B - 3d_C \\ 0 & -3 & 3 & 3d_A + 6d_C \end{array}\right) \\
&\longrightarrow \frac{1}{3}\left(\begin{array}{ccc|c} 1 & 1 & -2 & -3d_C \\ 0 & 1 & -1 & d_B - d_C \\ 0 & 0 & 0 & 3d_A + 3d_B + 3d_C \end{array}\right) \quad \leftarrow \text{note the third row.}
\end{aligned}$$

So there are no solutions unless $d_A = d_B = d_C = 0$, since $d_A \geq 0$, $d_B \geq 0$ and $d_C \geq 0$.

This is intuitive, since it costs \$1 to make \$1 of any product. *So no (nontrivial) external demand can be satisfied.*

Now, if the external demand is zero (i.e. $d_a = d_B = d_C = 0$), then

$$\left(\begin{array}{ccc|c} 1 & 1 & -2 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right) \to \left(\begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

The first row says that $x_A = x_C$, and the second row says that $x_B = x_C$, hence $x_A = x_B = x_C$. This means that, when external demand is nonexistent, the three industries can produce at *any* level, supplying only each other, as long as their productions are equal.