Supplementary Questions for HP Chapter 6

1. Solve the following matrix equation for a, b, c and d:

$$\begin{bmatrix} a-b & b+c \\ 3d+c & 2a-4d \end{bmatrix} = \begin{bmatrix} 8 & 1 \\ 7 & 6 \end{bmatrix}$$

2. Two-Commodity Market Model

Consider a model in which only two commodities are related to each other.

For i equal to 1 or 2, let:

Q_{di} be the quantity demanded of the commodity i

Q_{si} be the quantity supplied of the commodity i

P_i is the price of commodity i.

For simplicity, the demand and supply functions of both commodities are assumed to be linear. Such a model can be written as:

$$\begin{aligned} Q_{d1} - Q_{s1} &= 0 \\ Q_{d1} &= a_0 + a_1 P_1 + a_2 P_2 \\ Q_{s1} &= b_0 + b_1 P_1 + b_2 P_2 \\ Q_{d2} - Q_{s2} &= 0 \\ Q_{d2} &= \alpha_0 + \alpha_1 P_1 + \alpha_2 P_2 \\ Q_{s2} &= \beta_0 + \beta_1 P_1 + \beta_2 P_2 \end{aligned}$$

where the a, b, α and β are (appropriately) chosen coefficients.

(a) Write the above system as a matrix equation consisting of column matrices, where each column matrix consists of the coefficients that are all associated with the same variable.

(b) Write the above system as a single matrix equation (i.e., in the form $AX = B$, where A is the coefficient matrix and X and B are column matrices).

3. Let $A = \begin{pmatrix} 6 & 9 & 0 \\ -4 & -6 & 0 \\ 1 & 3 & 1 \end{pmatrix}$. Find A^{65}.

4. (a) Express the equations

$$\begin{aligned} y_1 &= x_1 - x_2 + x_3 \\ y_2 &= 3x_1 + x_2 - 4x_3 \\ y_3 &= -2x_1 - 2x_2 + 3x_3 \end{aligned} \qquad \text{and} \qquad \begin{aligned} z_1 &= 4y_1 - y_2 + y_3 \\ z_2 &= -3y_1 + 5y_2 - y_3 \end{aligned}$$

in the matrix forms $Y = AX$ and $Z = BY$. Then use these to obtain a direct relationship $Z = CX$ between Z and X.

(b) Use the equation $Z = CX$ in (a) to express z_1 and z_2 in terms of x_1, x_2 and x_3.

(c) Check the result in (b) by directly substituting the equations for y_1, y_2 and y_3 into the equations for z_1 and z_2 and then simplifying.

5. Let $A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ and $X = \begin{bmatrix} x \\ y \end{bmatrix}$

(a) Find all scalars k such that $AX = kX$ has a non-zero solution X.

(b) For each value of k from part (a), find all non-zero X such that $AX = kX$.

6. A restaurant owner plans to use x tables seating four, y tables seating six, and z tables seating eight, for a total of 20 tables. When fully occupied, the tables seat 108 customers. If only half of the x tables, half of the y tables and one-fourth of the z tables are used, each fully occupied, then 46 customers will be seated. Find x, y, z.

7. The pattern of unemployment over a period of time can be described by a Markov model, the linear system

$$\begin{aligned} x_{t+1} &= qx_t + py_t \\ y_{t+1} &= (1-q)x_t + (1-p)y_t \end{aligned}$$

For any k, x_k denotes the percentage of people of working age employed at time k, and y_k denotes the percentage of people of working age who are unemployed at time k. p, q are constants such that $0 \leq p \leq 1$, $0 \leq q \leq 1$.

If the unemployment and employment numbers remain constant, this yields the linear system

$$\begin{aligned} x &= qx + py \\ y &= (1-q)x + (1-p)y \\ 1 &= x + y \end{aligned}$$

or

$$\begin{aligned} (q-1)x + py &= 0 \\ (1-q)x - py &= 0 \\ x + y &= 1 \end{aligned}$$

(a) If p and q lie between 0 and 1, how many solutions does this system have? Why?

(b) Ignoring the condition that p and q lie between 0 and 1, find values of p and q so that this system has no solutions.

8. A box containing pennies, nickels and dimes has 13 coins with a total value of 83 cents. How many coins of each type are in the box?

9. Solve the following system by the method of reduction:

$$\begin{array}{cccccccc} x_1 & +3x_2 & -2x_3 & & +2x_5 & & = & 0 \\ 2x_1 & +6x_2 & -5x_3 & -2x_4 & +4x_5 & -3x_6 & = & -1 \\ & & 5x_3 & +10x_4 & & +15x_6 & = & 5 \\ 2x_1 & +6x_2 & & +8x_4 & +4x_5 & +18x_6 & = & 6 \end{array}$$

10. Find the values(s) of the constant c for which the following system

$$x + 3y + 2z = 0$$
$$x + cy + 4z = 0$$
$$2y + cz = 0$$

has infinitely many solutions.

11. Without using row-reduction, determine whether the following matrices are invertible:
(a)

$$A = \begin{bmatrix} 2 & 1 & -3 & 1 \\ 0 & 5 & 4 & 3 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 3 \end{bmatrix}$$

(b)

$$B = \begin{bmatrix} 5 & 1 & 4 & 1 \\ 0 & 0 & 2 & -1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 7 \end{bmatrix}$$

Hint: Consider the associated homogeneous system of equations for each matrix. Also, note that if a system $AX = C$ (including $C = 0$) has a unique solution, then A is invertible.

12. (a) Let A be a square matrix. Show that
(i) $(I - A)^{-1} = I + A + A^2 + A^3$ if $A^4 = 0$
(ii) $(I - A)^{-1} = I + A + A^2 + A^3 + \cdots + A^{n-1}$ if $A^n = 0$
(b) Let J_n be the $n \times n$ matrix each of whose entries is 1. Show that

$$(I - J_n)^{-1} = I - \frac{1}{n-1} J_n.$$

13. Let I denote the $n \times n$ identity matrix and let A be an $n \times n$ matrix such that $A^3 - 3A^2 + 2A + I = 0$. Then A^{-1}
(a) does not exist
(b) equals $-A^3 + 3A^2 - 2A$

(c) equals $A(A-I)(A-2I)$
(d) equals $(I-A)(A-2I)$
(e) equals $A^2-3A+2I$

14. A 3-industry sector of an economy is composed of industries A, B and C. The external demands for their products are d_A, d_B and d_C respectively.

(a) Suppose that each dollar of output in any of the industries requires a quarter (25 cents) of output from that industry itself and a quarter (25 cents) from each of the other industries. If external demand is to be satisfied exactly, find the total production of each industry (in terms of d_A, d_B and d_C). Show the technology matrix and the Leontiff matrix in solving this problem.

(b) As in (a), but each dollar of output of any industry requires $\frac{1}{3}$ of a dollar of input from that industry itself and $\frac{1}{3}$ of a dollar from each of the other industries, give all possible answers and interpret the result.