Solutions to Supplementary Questions for HP Chapter 5 and Sections 1 and 2 of the Supplementary Material

1. (a) Let P be the recommended retail price of the toy. Then the retailer may purchase the toy at prices of $0.7P$ and $0.75P$ by cash and credit respectively. The interest rate over the six months is given by

$$0.7P(1+i) = 0.75P$$
$$i = \frac{0.75}{.7} - 1 \qquad \text{for six months.}$$

We convert this to an effective rate of interest by the equation

$$r_e = (1+i)^2 - 1 = \left(\frac{0.75}{.7}\right)^2 - 1 = 0.148, \quad \text{so } 14.8\%$$

(b) As in (a), the interest rate over three months is given by $0.7(1+i) = 0.725$ and $r_e = (1+i)^4 - 1$ so $(1+i) = (r_e+1)^{\frac{1}{4}}$ and $0.7(r_e+1)^{\frac{1}{4}} = .725$, $r_e = \left(\frac{0.725}{0.7}\right)^4 - 1 = 0.1507$, so 15.07% which is a slightly higher effective rate of interest than (a).

2. (a) The monthly interest rate r is given by

$$1.09 = (1+r)^{12}$$
$$r = (1.09)^{\frac{1}{12}} - 1 = 0.007207323.$$

So the initial monthly payment X is given by

$$19750 = X a_{\overline{240}|r}$$
$$X = \frac{19750[(1.09)^{\frac{1}{12}} - 1]}{1 - (1.09)^{-\frac{240}{12}}} = \$173.26.$$

The present value of the 153 remaining payments immediately after the 87th payment is

$$173.26 a_{\overline{153}|r} = 173.26 \left[\frac{1 - (1.09)^{-\frac{153}{12}}}{(1.09)^{\frac{1}{12}} - 1}\right]$$
$$= \$16,027.52.$$

An effective rate of 10% corresponds to a monthly rate of

$$i = (1.10)^{\frac{1}{12}} - 1 = 0.00797414$$

Thus the value of the revised payments would be Y where

$$16027.52 = Y a_{\overline{153}|i}$$

$$Y = 16027.52 \left[\frac{(1.10)^{\frac{1}{12}} - 1}{1 - (1.10)^{-\frac{153}{12}}} \right] = \$181.71$$

which is an increase of $181.71 - 173.26 = \$8.45$.

(b) Given the present value of $\$16,027.52$ of the remaining 153 payments immediately after payment 87, we wish to solve for n where

$$16027.52 = (173.26)a\,\overline{_{n}|}_i = 173.26 \left[\frac{1 - (1.10)^{-\frac{n}{12}}}{(1.10)^{\frac{1}{12}} - 1} \right]$$

$$1 - \frac{16027.52}{173.26}[(1.10)^{\frac{1}{12}} - 1] = 1.10^{-\frac{n}{12}}$$

$$-\frac{n}{12}\ln 1.10 = \ln\left\{ 1 - \frac{16027.52}{173.26}[(1.10)^{\frac{1}{12}} - 1] \right\}$$

$$n = \frac{-12\ln\left\{ 1 - \frac{16027.52}{173.26}[(1.10)^{\frac{1}{12}} - 1] \right\}}{\ln 1.10}$$

$$= 168.47$$

Since n must be an integer, there remain 169 payments. The final payment is given by

$$16027.52 = 173.26\, a\,\overline{_{168}|}_i + X(1+i)^{-169}$$

where $(1+i)^{12} = 1.10$

$$X = (16027.52 - 173.26\, a\,\overline{_{168}|}_i)(1.10)^{\frac{169}{12}} = \$81.83$$

3. This annuity may be considered as the sum of 20 annuities of equal payments each year. The first one is payments of $\$2,000$ for 20 years, the second one is payments of $\$100$ for 19 years, beginning one year after the first, ..., the kth annuity is payments of $\$100$ for $20 - k + 1$ years beginning $k - 1$ years after the first annuity.

The purchase price P of the original annuity is the sum of the present values of the

20 annuities at the time of purchase.

$$P = 2000a_{\overline{20}|0.04} + \sum_{k=2}^{20} 100a_{\overline{20-k+1}|.04}(1.04)^{-k+1}$$

$$= 2000a_{\overline{20}|0.04} + 100\sum_{k=1}^{19} a_{\overline{20-k}|.04}(1.04)^{-k}$$

$$= 2000a_{\overline{20}|0.04} + 100\sum_{k=1}^{19}\left(\frac{1-(1.04)^{k-20}}{0.04}\right)(1.04)^{-k}$$

$$= 2000a_{\overline{20}|0.04} + \frac{100}{0.04}\sum_{k=1}^{19}\left((1.04)^{-k}-(1.04)^{-20}\right)$$

$$= 2000a_{\overline{20}|0.04} + \frac{100}{0.04}\left(\sum_{k=1}^{19}(1.04)^{-k}\right) - \frac{100}{(0.04)(1.04)^{20}}\sum_{k=1}^{19}1$$

$$= 2000a_{\overline{20}|0.04} + \frac{100}{1.04(.04)}\sum_{k=0}^{18}\left(\frac{1}{1.04}\right)^{k} - \frac{(19)(100)}{(0.04)(1.04)^{20}}$$

$$= 2000a_{\overline{20}|0.04} + \frac{100}{1.04(.04)}\left(\frac{1-\left(\frac{1}{1.04}\right)^{19}}{1-\frac{1}{1.04}}\right) - \frac{1900}{(0.04)(1.04)^{20}}$$

by the formula for the sum of a geometric series

$$= 2000a_{\overline{20}|0.04} + \frac{100}{1.04(.04)}\left(\frac{\frac{(1.04)^{19}-1}{(1.04)^{19}}}{\frac{0.04}{1.04}}\right) - \frac{1900}{(0.04)(1.04)^{20}}$$

$$= 2000\left(\frac{1-(1.04)^{-20}}{0.04}\right) - \frac{1900}{(0.04)(1.04)^{20}} + \frac{(100)((1.04)^{19}-1)}{(0.04)^2(1.04)^{19}}$$

$$= \$38,337.12$$

The purchase price of the original annuity is $38,337.12.

4. Interest rates corresponding to 8% compounded annually are:

monthly:	$(1.08)^{\frac{1}{12}}-1 = 0.0064340$
quarterly:	$(1.08)^{\frac{1}{4}}-1 = 0.0194265$
semi-annually:	$(1.08)^{\frac{1}{2}}-1 = 0.0392305.$

If X is the amount of the semi-annual payments of the revised annuity, the equation of value at the time of the request on November 1, 1985 is:

$$Xa_{\overline{43}|0.0392}(1.0064340)^3 = 200a_{\overline{22}|0.08}(1.0064340)^9 + 80a_{\overline{65}|0.0194265}(1.0064340)$$

$$+ 15a_{\overline{225}|0.0064340}$$

i.e. the annual annuity has a present value of $200a_{\overline{22}|0.08}$ nine months before November 1, 1985, and so on.

Solving for X, we get $X = \$328.50$ to the nearest half dollar. (Your answer may vary by a few pennies depending on rounding errors.) The revised annuity has semi-annual payments of $328.50.

5. The interest payments on the loan form an annuity of equal payments of $\frac{0.055}{2}(20000) = \$550$ at semi-annual periods.

With a nominal interest rate of 6% compounded semi-annually, the equation of value at the beginning of the seven years is

$$23362.67(1.03)^{-n} = 550a_{\overline{14}|0.03} + 22000(1.03)^{-14}$$

where n is the number of half years before the repayment of $\$23,362.67$ is made.

$$(1.03)^{-n} = \frac{550a_{\overline{14}|0.03} + 22000(1.03)^{-14}}{23362.67}$$

$$-n\ln(1.03) = \ln\left(\frac{550a_{\overline{14}|0.03} + 22000(1.03)^{-14}}{23362.67}\right)$$

$$n = \frac{\ln(23362.67) - \ln(550a_{\overline{14}|0.03} + 22000(1.03)^{-14})}{\ln(1.03)}$$

$$n = 4 \text{ half years } = 2 \text{ years}$$

6. Interest rates corresponding to an effective rate of 10% are:

    monthly: $(1.1)^{\frac{1}{12}} - 1 = 0.00797$

    quarterly: $(1.1)^{\frac{1}{4}} - 1 = 0.02411$

    semi-annually: $(1.1)^{\frac{1}{2}} - 1 = 0.04881$

If the initial payments are X dollars, the equation of value when the man receives the annuity is

$$2049 = Xa_{\overline{10}|0.04881} + 2Xa_{\overline{60}|0.00797}(1.1)^{-5} + 4Xa_{\overline{20}|0.02411}(1.1)^{-10}$$

$$2049 = X\left[a_{\overline{10}|0.04881} + 2(1.1)^{-5}a_{\overline{60}|0.00797} + 4(1.1)^{-10}a_{\overline{20}|0.02411}\right]$$

$$X = 2049\left[a_{\overline{10}|0.04881} + 2(1.1)^{-5}a_{\overline{60}|0.00797} + 4(1.1)^{-10}a_{\overline{20}|0.02411}\right]^{-1}$$

$$X = 2049\left[\frac{1 - (1.04881)^{-10}}{0.04881} + 2(1.1)^{-5}\frac{(1 - (1.00797)^{-60})}{0.00797}\right.$$

$$\left. +4(1.1)^{-10}\frac{(1 - (1.02411)^{-20})}{0.02411}\right]^{-1}$$

$$X = 22.50$$

7. (a) We have $16,000 = Ra_{\overline{10}|.08}$

$$\Rightarrow R = \frac{16,000}{\left(\frac{1-(1.08)^{-10}}{.08}\right)} = \$2,384.47$$

b) The loan outstanding just after the fourth payment is:

$$\$2,384.47a_{\overline{6}|.08}$$

$$= \$2,384.47\left(\frac{1-(1.08)^{-6}}{.08}\right)$$

$$= \$11,023.12$$

since there are still six annual payments of $\$2,384.47$ to be made—that is, before the change in interest. So the revised installment R' is

$$R' = \frac{11,023.12}{a_{\overline{6}|.10}} = \frac{11,023.12}{\left(\frac{1-(1.10)^{-6}}{.10}\right)} = \$2,530.99$$

c) The loan outstanding just after the seventh payment is made is

$$\$2,530.99a_{\overline{3}|.10} = \$2,530.99\left(\frac{1-(1.10)^{-3}}{.10}\right)$$

$$= \$6,294.20$$

So the revised installment R'' is

$$R'' = \frac{6,294.20}{a_{\overline{3}|.09}} = \frac{6,294.20}{\left(\frac{1-(1.09)^{-3}}{.09}\right)} = \$2,486.55$$

We now find the effective rate of interest r paid by the borrower on the completed transaction:

The present value of the first four installments is $\$2,384.47a_{\overline{4}|r}$.

Also, the value at year four of the next three installments is $\$2,530.99a_{\overline{3}|r}$. Hence the present value of this is $\$2,530.99a_{\overline{3}|r}(1+r)^{-4}$.

And the value at year seven of the final three installments is $\$2,486.55a_{\overline{3}|r}$. Hence the present value of this is $\$2,486.55a_{\overline{3}|r}(1+r)^{-7}$.

The sum of these is of course $\$16,000$:

$$16,000 = 2,384.47\left(\frac{1-(1+r)^{-4}}{r}\right) + 2,530.99\left(\frac{1-(1+r)^{-3}}{r}\right)(1+r)^{-4}$$

A

$$+ 2,486.55\left(\frac{1-(1+r)^{-3}}{r}\right)(1+r)^{-7}$$

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We now guess by interpolation. r should be somewhat less than 9%, say 8.8%?

With $r = .088$, the RHS of **A** is \$15,847.68, so our guess for r was too large...

With $r = .086$, the RHS of **A** is \$15,990.49

With $r = .0859$, the RHS of **A** is \$15,997.68

With $r = .08586$, the RHS of **A** is \$16,000.57

Hence, the actual correct r lies between 8.59% and 8.586%. Thus, to within one-hundredth of a percentage point, r is 8.59%.

8. (a) The annuity payments are shown as follows:

September 1, 1990	—
September 1, 1991	1000
January 1, 1992	$1000(1.05)$
May 1, 1992	$1000(1.05)^2$
$\vdots$	$\vdots$
May 1, 1996	$1000(1.05)^{14}$

We calculate the present value of the annuity on September 1, 1990. Summing up the present values of the payments, we have

$$1000(1.02)^{-3}\left[1 + (1.05)(1.02)^{-1} + (1.05)^2(1.02)^{-2} + \cdots + (1.05)^{14}(1.02)^{-14}\right]$$

$$= 1000(1.02)^{-3}\left[1 + \frac{1.05}{1.02} + \left(\frac{1.05}{1.02}\right)^2 + \cdots + \left(\frac{1.05}{1.02}\right)^{14}\right]$$

$$= \frac{1000}{(1.02)^3}\left[\frac{1 - \left(\frac{1.05}{1.02}\right)^{15}}{1 - \left(\frac{1.05}{1.02}\right)}\right]$$

by the formula for the sum of a geometric series

$$= 17450.80$$

The purchase price was \$17,450.80.

(b) The present value of the remaining payments just after the sixth payment is

$$\underbrace{(17450.80)(1.02)^9}_{\substack{\text{The present value of all}\\\text{payments at this time}}}$$

$$\underbrace{-1000[(1.02)^5 + (1.05)(1.02)^4 + \cdots + (1.05)^4(1.02) + (1.05)^5]}_{\substack{\text{the present value of payments}\\\text{1,2,...,6 at this time}}}$$

$$= (17450.80)(1.02)^9 - (1000)(1.02)^5 \left[1 + \frac{1.05}{1.02} + \cdots + \left(\frac{1.05}{1.02}\right)^5\right]$$

$$= (17450.80)(1.02)^9 - (1000)(1.02)^5 \left(\frac{1 - \left(\frac{1.05}{1.02}\right)^6}{1 - \left(\frac{1.05}{1.02}\right)}\right)$$

by the formula for the sum of a geometric series

$$= 13,724.21$$

Therefore the interest content of the seventh installment was $(0.02)(13,724.21)$ $= \$274.48$.

9. a) First, convert to annual interest compounded monthly:

$$\left(1 + \frac{r_m}{12}\right)^{12} = 1 + r_e = 1.07$$

$$\Rightarrow 1 + \frac{r_m}{12} = \sqrt[12]{1.07} \approx 1.005654145$$

$$\Rightarrow r_m = 12(\sqrt[12]{1.07} - 1) \approx 6.7849744, \%$$

$n = 12 \times 25 = 300$. So,

$$\$9,880 = Ra_{\overline{300}|.005654145}$$

$$\Rightarrow R = \frac{\$9,880}{\left(\frac{1-(1.07)^{-25}}{.005654145}\right)} = \$68.48.$$

b) The March 10th payment is the $13(12) + 8 = 164^{\text{th}}$ payment. So there are 136 payments remaining. The principal outstanding is

$$68.48a_{\overline{136}|.005654145} = \$6485.72.$$

c) The October 10th, 1989 payment is the $11(12) + 3 = 135^{\text{th}}$ payment. So, the principal contained is $(k = 135)$

$$R\left[1 - ra_{\overline{n-k+1}|,005654145}\right] = \$68.48[(1.005654145)^{-166}]$$

$$= \$68.48(.392215214)$$

$$= \$26.86$$

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d) April 10th, 1996 is the $17(12) + 9 = 213^{\text{th}}$ payment. March 10th, 1997 is the $213 + 11 = 224^{\text{th}}$ payment.

i) We look for the loan outstanding at the beginning of the 213th period and subtract the loan outstanding at the end of the 224th, or beginning of the 225th, period.

$$= \$68.48 \left\{ \left[\frac{1 - (1.005654145)^{-88}}{.005654145} \right] - \left[\frac{1 - (1.005654145)^{-76}}{.005654145} \right] \right\}$$

$$= \$68.48 \left[\frac{(1.005654145)^{-76} - (1.005654145)^{-88}}{.005654145} \right]$$

$$= \$68.48[7.537894] = \$516.19$$

ii) The total interest in these installments is just, by part (i)

$$12(\$68.48) - \$516.19 = \$305.57$$

10. a) We determine the contributions needed from i) and ii) individually and then sum them.

i) There is a $.0575(1,000,000) = \$57,500$ payment halfway through the year and another $\$57,500$ payment at the end of the year. We calculate the sum of the present values of these two payments (i.e., this is an annuity with two payments of $\$57,500$ semi-annually with 5% interest during each payment period).

$$PV = \$57,500 a_{\overline{2}|.05} = 57,500 \left(\frac{1 - (1.05)^{-2}}{.05} \right) = \$106,916.0998$$

So $\$106,916.10$ should be deposited at the start of the year to cover part (i).

ii) We determine the effective annual rate (compounded annually) of 10% compounded semi-annually

$$1 + r_a = \left(1 + \frac{r_s}{2}\right)^2 = (1.05)^2 = 1.1025.$$

So the effective annual rate (compounded annually) is 10.25%.

Now we consider the amount R of yearly payments needed so that the future value of a 20 year annuity *due* at $10\frac{1}{4}\%$ is $\$1,000,000$ (notice the payments of R are at the *start* of the year).

$$R = \frac{1,000,000}{s_{\overline{21}|.1025} - 1} = \frac{1,000,000}{\left(\frac{(1.1025)^{21} - 1}{.1025} \right) - 1} = \$15,392.49922$$

So $\$15,392.50$ should be deposited at the start of the year to cover part (ii).

The sum of the payments for (i) and (ii) is $\$122,308.60$, and this is the amount deposited by the company at the start of the year.

Note: a trickier method for solving (a):

We first pretend that, instead of making one payment at the start of the year, the company makes equal payments at months six and twelve in the year. Each payment will just be

$$\$57,500 + \frac{\$1,000,000}{s\,\overline{_{40|}}.05} = \$57,500 + \left(\frac{\$1,000,000}{\left(\frac{(1.05)^{40}-1}{.05}\right)}\right)$$

$$= \$57,500 + 8,278.16116$$

$$= \$65,778.16$$

paid semi-annually.

But now, instead of $\$65,778.16$ paid at months six and twelve of the year, we calculate the equivalent lump sum payment at the beginning of the year:

$$\$65,778.16116a\,\overline{_{2|}}.05 = \$65,778.16116\left(\frac{1-(1.05)^{-2}}{.05}\right) = \$122,308.60$$

b) We answer ii) first. Since the 'coupon payments' and the 'face value payment at redemption' do not change, then neither does the corporation change how much they deposit at the start of the year. They still must cover the same amount. Rather, the company is going to sell the bonds at a discount in order to make the purchase of the bonds worthwhile for those investors who want a yield of 12% (payable semi-annually).

Now, the yield rate i desired is 6% per period (half-year). The coupon rate r is 5.75% per period. $n = 20 \times 2 = 40$; and the face value is $\$1,000,000$. We calculate the price P of the bond:

$$P = V(1+i)^{-n} + rVa\,\overline{_{n|}}i$$

$$= \$1,000,000(1.06)^{-40} + .0575(\$1,000,000)\left(\frac{1-(1.06)^{-40}}{.06}\right)$$

$$= 97,222.18771 + 865,162.0701$$

$$= \$962,384.26$$

11. First, we look for the quarterly yield rate. Using the formula of the 'bond salesman's method', we guess $i \simeq \frac{r-\frac{p-V}{nV}}{1+\frac{p-V}{2V}}$. Here, $n = 80$, $V = 100$, $P = 49.50$, $r = .01375$. (Note: for the 'bond salesman's method' see Supplementary Notes, Prob. 7 on page 10.) $\simeq \frac{.01375+.0063125}{1-.2575} = .0268$. Substituting $i = .0268$ in $100(1+i)^{-80} + 1.375\left(\frac{1-(1+i)^{-80}}{i}\right)$ we get $P = \$57.18$ (awful estimate!), so we guess $i = .03$. Now, we get $P = \$50.92$

Setting $i = .0307$, $P = \$49.70$

Setting $i = .03083$, $P = \$49.48$

Setting $i = .03081$, $P = \$49.5149$

Setting $i = .030813$, $P = \$49.5098$

So the quarterly yield is between 3.0813% and 3.082%.

b) The investor sold the bonds for $\$12,521.62$. Moreover, the investor had already received four bond payments of $\$275$. The present value of these payments is $275 s_{\overline{4}|.0308}$ $= \$1,151.87$. So, overall, the investor receives $\$1,151.87 + \$12,521.62 = \$13,673.49$ from the bonds.

For the mortgage, we first calculate the periodic monthly rate

$$1 + r_m = (1.06)^{\frac{1}{6}} = 1.009759 \Rightarrow r_m \approx .9759\%$$

Now, the mortgage will give, after one year

$$\$1,100 s_{\overline{12}|.009759} = 1100(12.665514)$$
$$= \$13,932.07$$

This is more than the money made from the bonds, so the bonds were not the right choice.