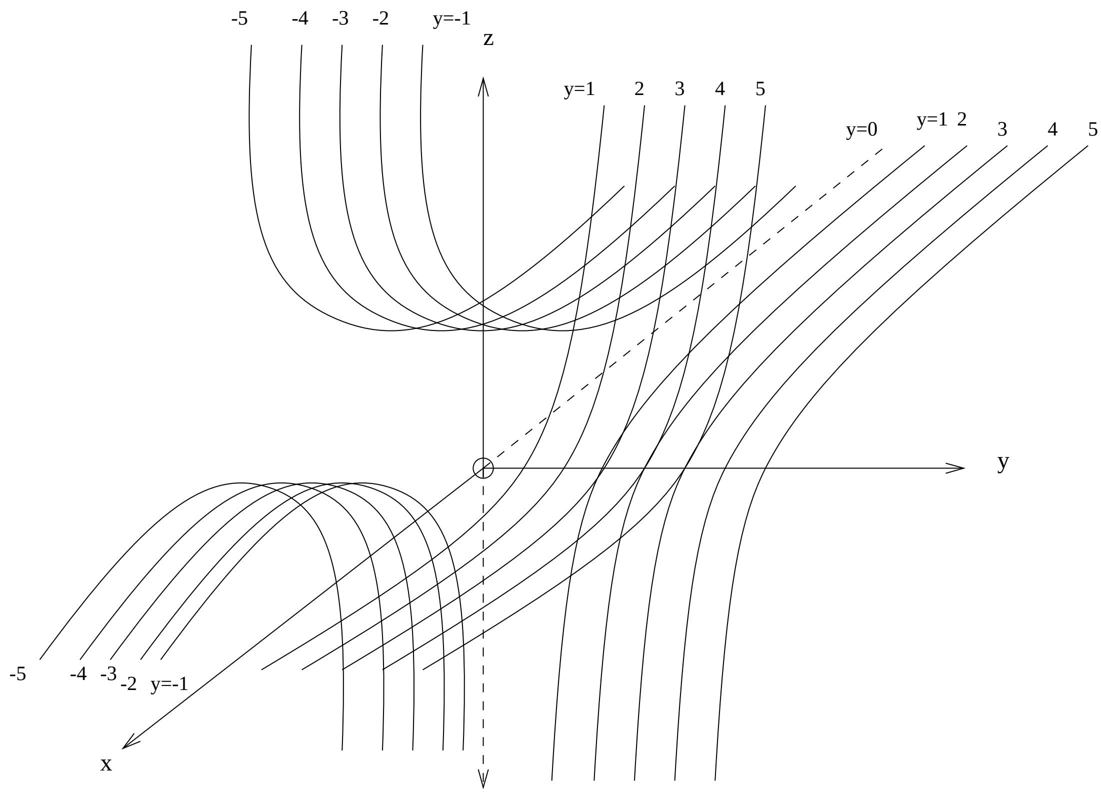
Solutions to Supplementary Questions for HP Chapter 17

1. Graphing will be done by taking "slices", as was alluded to in HP, pg. 957, example 6.

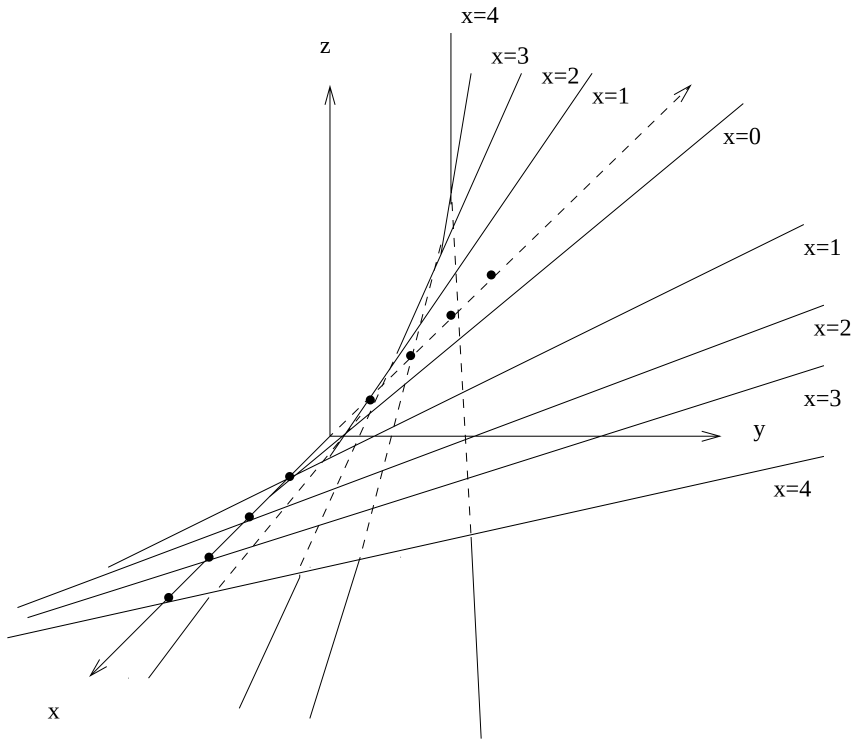
(a) It is best to look at this as $y = xz$, but remembering that $x \neq 0$. Now take slices by fixing various values of y:

$$y = 0 \Rightarrow z = 0 \text{ for all } x$$
$$y = 1 \Rightarrow xz = 1$$
$$y = 2 \Rightarrow xz = 2$$
$$y = -1 \Rightarrow xz = -1$$



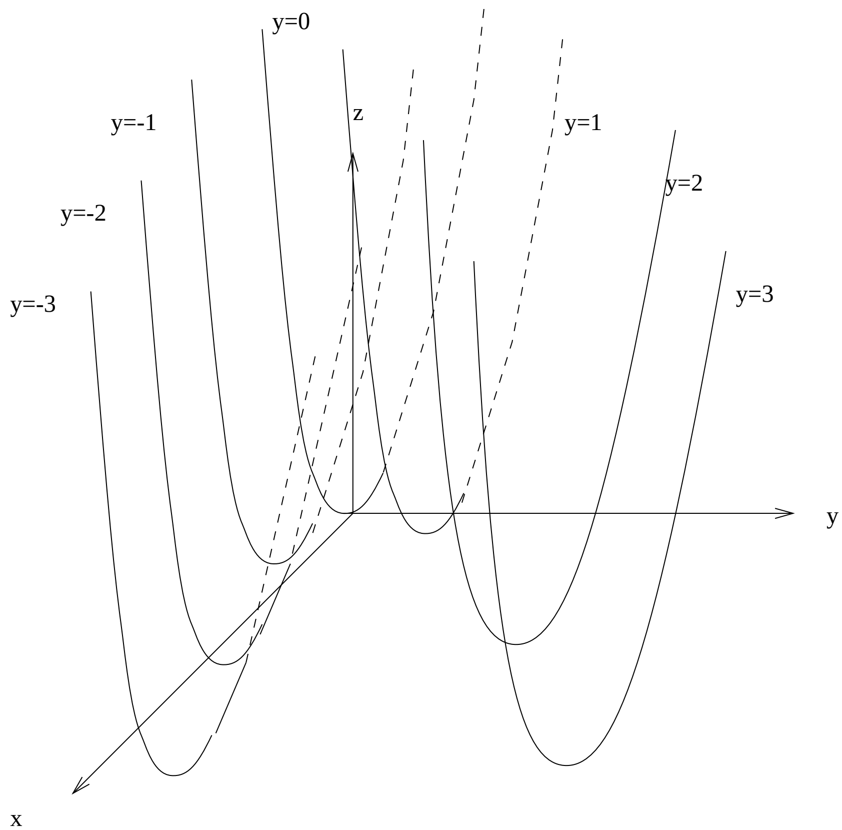
(b) We slice along the x-axis:

$$x = -2 \Rightarrow z = e^2 y$$
$$x = -1 \Rightarrow z = ey$$
$$x = 0 \Rightarrow z = y$$
$$x = 1 \Rightarrow z = \frac{y}{e}$$
$$x = 2 \Rightarrow z = \frac{y}{e^2}$$



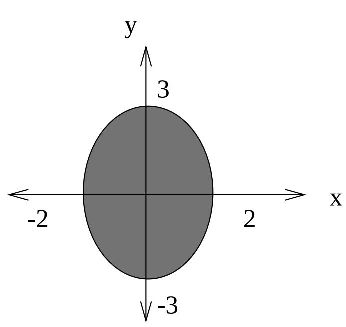
(c) $z = x^2 - y^2$ We slice along the y axis

$$y = -2 \Rightarrow z = x^2 - 4$$
$$y = -1 \Rightarrow z = x^2 - 1$$
$$y = 0 \Rightarrow z = x^2$$
$$y = 1 \Rightarrow z = x^2 - 1$$
$$y = 2 \Rightarrow z = x^2 - 4$$



2. (a) Domain:

We need $36 - 9x^2 - 4y^2 \geq 0 \Rightarrow 9x^2 + 4y^2 \leq 36$.
$\Rightarrow \frac{x^2}{2^2} + \frac{y^2}{3^2} \leq 1$. So the domain is everything inside of (and on) the ellipse $\frac{x^2}{2^2} + \frac{y^2}{3^2} = 1$.
Domain:



Range: $x^2, y^2 \geq 0 \;\Rightarrow\; 9x^2, 4y^2 \geq 0 \;\Rightarrow\; -9x^2 - 4y^2 \leq 0,$

$$\Rightarrow\; 36 - 9x^2 - 4y^2 leq 36$$
$$\Rightarrow\; \sqrt{36 - 9x^2 - 4y^2} \;\leq\; 6.$$

Also, if $x = 0$ and $y = 3$, then $\sqrt{36 - 9x^2 - 4y^2} = 0$. Since no square root is negative, then it follows that the range is [0,6].

b) $x^2 \ln(x - y + z)$

Domain: We need $x - y + z > 0$ or $x + z > y$.

Range:

i) Set $y = 0, z = 0$. Now, for $x > 0$,

$$\lim_{x\to\infty} x^2 = \infty \quad \text{and} \quad \lim_{x\to\infty} \ln(x - y + z) = \lim_{x\to\infty} \ln(x) = \infty \quad \text{so}$$
$$\lim_{x\to\infty} x^2 \ln(x - y + z) = \infty.$$

ii) Set $x = 1, z = 0$.

Now $\lim_{y\to 1^-} \ln(x-y+z) = \lim_{y\to 1^-} \ln(1-y) = -\infty$, hence $\lim_{y\to 1^-} x^2 \ln(x-y+z) = -\infty$. So the range is $(-\infty, \infty)$

c) $\frac{x}{yz}$. Domain: We need $yz \neq 0$. So $y \neq 0$ and $z \neq 0$.
Range: Set $y = 1, z = 1$. Now it is obvious that the range is $(-\infty, \infty)$.

d) $\frac{1}{\sqrt{x^2+y^2+z^2-1}}$ Domain: Here, we need $x^2 + y^2 + z^2 - 1 > 0$, so $x^2 + y^2 + z^2 > 1$. This is everything strictly outside of the unit radius sphere.
Range: Obviously the range is a subset of $(0, \infty)$.
Now, set $y = 0, z = 0$. Here, $\lim_{x\to\infty} \frac{1}{\sqrt{x^2+y^2+z^2-1}} = \lim_{x\to\infty} \frac{1}{\sqrt{x^2-1}} = 0$.
Also, $\lim_{x\to 1^+} \frac{1}{\sqrt{x^2+y^2+z^2-1}} = \lim_{x\to 1^+} \frac{1}{\sqrt{x^2-1}} = \infty$. So the range is $(0, \infty)$.

3.

$$PV = nRT \Rightarrow P = \frac{nRT}{V} \Rightarrow \frac{\partial P}{\partial V} = -\frac{nRT}{V^2}$$

$$PV = nRT \Rightarrow V = \frac{nRT}{P} \Rightarrow \frac{\partial V}{\partial T} = \frac{nR}{P}$$

$$PV = nRT \Rightarrow T = \frac{PV}{nR} \Rightarrow \frac{\partial T}{\partial P} = \frac{V}{nR}$$

Now, $\frac{\partial P}{\partial V} \cdot \frac{\partial V}{\partial T} \cdot \frac{\partial T}{\partial P} = \frac{-nRT}{V^2} \cdot \frac{nR}{P} \cdot \frac{V}{nR} = -\frac{nRT}{PV}$.
But $PV = nRT \Rightarrow \frac{nRT}{PV} = 1$, so $-\frac{nRT}{PV} = -1$.

4.

For the xy-plane, $z = 0$, and so $\frac{\partial z}{\partial x} = \frac{\partial z}{\partial y} = 0$. Hence, for any point $f(x_0, y_0)$ to have its tangent plane horizontal (i.e. parallel to the surface) then $f_x(x_0, y_0) = 0$ and $f_y(x_0, y_0) = 0$ as well. We find $f_x(x, y)$ and $f_y(x, y)$:

i) $\frac{\partial}{\partial x}(f(x,y)) = 2x + 2y - 6$. Setting this to zero gives $x + y = 3$.

ii) $\frac{\partial}{\partial y}(f(x,y)) = 2x + 10$. Setting this to zero gives $x = -5$.

From (i) $y = 3 - x$, so $y = 3 - (-5) = 8$

So the only point at which the tangent plane to the surface $f(x,y) = x^2 + 2xy + 10y - 6x$ is horizontal, is $(x,y) = (-5, 8)$.

5.

(a) i) $f_x(x,y)$. Treating y as a constant, by the Fundamental Theorem of Calculus,

$$\int_x^y e^{t^2}\,dt = F(y) - F(x), \quad \text{where} \quad \frac{d}{dx}(F(x)) = e^{x^2}.$$

so $f_x(x,y) = \frac{d}{dx}(\int_x^y e^{t^2}\,dt) = \frac{d}{dx}(F(y) - F(x)) = -e^{x^2}$

ii) $f_y(x,y): \frac{d}{dy}(\int_x^y e^{t^2}\,dt) = \frac{d}{dy}(F(y) - F(x)) = e^{y^2} (= f_y(x,y))$

(b) i) $f_x(x,y): \frac{d}{dx}(\int_y^x \frac{e^t}{t}\,dt) = \frac{e^x}{x}$

ii) $f_y(x,y): \frac{d}{dy}(\int_y^x \frac{e^t}{t}\,dt) = -\frac{e^y}{y}$

6.

Let $f(x,y,z) = x^3\sqrt{y^2+z^2}$ so

$$\frac{\partial f}{\partial x} = 3x^2\sqrt{y^2+z^2}, \quad \frac{\partial f}{\partial y} = \frac{x^3 y}{\sqrt{y^2+z^2}}, \quad \frac{\partial f}{\partial z} = \frac{x^3 z}{\sqrt{y^2+z^2}}.$$

Then

$$\begin{aligned}
(1.98)^3\sqrt{(3.01)^2+(3.97)^2} &= f(1.98, 3.01, 3.97) \\
&= f(2-0.02, 3+0.01, 4-0.03) \\
&\approx f(2,3,4) + \frac{\partial f}{\partial x}(2,3,4)(-0.02) + \frac{\partial f}{\partial y}(2,3,4)(0.01) \\
&\quad + \frac{\partial f}{\partial z}(2,3,4)(-0,03) \qquad \text{so that} \quad \begin{aligned} dx_1 &= -0.02 \\ dx_2 &= 0.01 \\ dx_3 &= -0.03 \end{aligned} \\
&= 2^3\sqrt{3^2+4^2} + (-0.02)(3)(2)^2\sqrt{3^2+4^2} \\
&\quad + (0.01)\frac{(2)^3(3)}{\sqrt{3^2+4^2}} + (-0.03)\frac{(2)^3(4)}{\sqrt{3^2+4^2}} \\
&= 8(5) - 0.02(12)(5) + 0.01(24/5) - 0.03(32/5) \\
&= 40 - 0.02(60) + \frac{1}{5}[0.24 - 0.96] \\
&= 40 - 1.2 + 0.2(-0.72) = 40 - 1.2 - 0.144 \\
&= 40 - 1.344 = 38.656
\end{aligned}$$

7.

(i) $\frac{\partial Q_1}{\partial P_1} = a_{11}K_1P_1^{a_{11}-1}P_2^{a_{12}}I^{b_1}$. So $\varepsilon_1 = \frac{P_1\frac{\partial Q_1}{\partial P_1}}{Q_1} = \frac{P_1(a_{11}K_1P_1^{a_{11}-1},P_2^{a_{12}}I^{b_1})}{K_1P_1^{a_{11}}P_2^{a_{12}}I^{b_1}} = a_{11}$

(ii) $\frac{\partial Q_1}{\partial P_2} = a_{12}K_1P_1^{a_{11}}P_2^{a_{12}-1}I^{b_1}$. So $\varepsilon_{Q_1,P_2} = \frac{P_2(a_{12}K_1P_1^{a_{11}}P^{a_{12}-1}I^{b_1})}{K_1P_1^{a_{11}}P_2^{a_{12}}I^{b_1}} = a_{12}$

(iii) $\frac{\partial Q_1}{\partial I} = b_1K_1P_1^{a_{11}}P_1^{a_{12}}I^{b_1-1}$ So $\varepsilon_{Q_1,I} = \frac{I(b_1K_1P_1^{a_{11}}P_2^{a_{12}}I^{b_1-1})}{K_1P_1^{a_{11}}P_2^{a_{12}}I^{b_1}} = b_1$

8. (a) $f(27,16,64) = 10(27)^{\frac{1}{2}}(16)^{\frac{1}{2}}(64)^{\frac{1}{6}} = 10(3)(4)(2) = 240$

(b)

i) $f_{x_1}(x_1,x_2,x_3) = \frac{10}{3}x_1^{-\frac{2}{3}}x_2^{\frac{1}{2}}x_3^{\frac{1}{6}}$
$\Rightarrow f_{x_1}(27,16,64) = \frac{10}{3}(\frac{1}{9})(4)(2) = \frac{80}{27}$

ii) $f_{x_2}(x_1,x_2,x_3) = \frac{10}{2}x_1^{\frac{1}{3}}x_2^{-\frac{1}{2}}x_3^{\frac{1}{6}}$
$\Rightarrow f_{x_2}(27,16,64) = 5(3)(\frac{1}{4})(2) = \frac{15}{2}$

iii) $f_{x_3}(x_1,x_2,x_3) = \frac{10}{6}x_1^{\frac{1}{3}}x_2^{\frac{1}{2}}x_3^{-\frac{5}{6}}$
$\Rightarrow f_{x_3}(27,16,64) = \frac{5}{3}(3)(4)(\frac{1}{32}) = \frac{5}{8}$

Now, x_1 increases by .1 ($\Delta x_1 = .1$) x_2 decreases by .3 ($\Delta x_2 = -.3$) and $\Delta x_3 = 0$. So

$$\begin{aligned} f(27.1,15.7,64) &\approx f(27,16,64) + \sum_{i=1}^{3}\Delta x_i f x_i(27,16,64) \\ &= 240 + (.1)(\frac{80}{27}) - .3(\frac{15}{2}) + 0(\frac{5}{8}) \\ &= 238.0463 \end{aligned}$$

(c) $f(27.1,15.7,64) = 10(27.1)^{\frac{1}{3}}(15.7)^{\frac{1}{2}}(64)^{\frac{1}{6}} = 238.0325.$

(d) Here $\Delta x_1 = \Delta x_2 = 0.2$ and $\Delta x_3 = -0.4$

$$\begin{aligned} f(27.2,16.2,63.6) &\approx f(27,16,64) + \sum_{i=1}^{3}\Delta x_i f x_i(27,16,64) \\ &= 240 + (.2)(\frac{80}{27}) + .2(\frac{15}{2}) - .4(\frac{5}{8}) \\ &= 241.8426 \end{aligned}$$

By calculator, $f(27.2,16.2,63.6) = (27.2)^{\frac{1}{2}}(16.2)^{\frac{1}{2}}(63.6)^{\frac{1}{6}} = 241.8373.$

9.

(a)

$$\begin{aligned} \frac{\partial P}{\partial n} &= V(-\ln(1+i))(1+i)^{-n} + \frac{rV}{i}(\ln(1+i)(1+i)^{-n}) \\ &= V\big(\ln(1+i)\big)(1+i)^{-n}\left(\frac{r}{i}-1\right) \end{aligned}$$

Since $\ln(1+i) > 0$ and hence $V\ln(1+i)(1+i)^{-n} > 0$, then:

i) If $i > r$, then $\left(\frac{r}{i} - 1\right) < 0$, hence $\frac{\partial P}{\partial n} < 0$ and P increases as n decreases.
ii) If $i < r$, then $\left(\frac{r}{i} - 1\right) > 0$, hence $\frac{\partial P}{\partial n} > 0$ and P decreases as n decreases.
iii) If $i = r$, then $\left(\frac{r}{i} - 1\right) = 0$, hence $\frac{\partial P}{\partial n} = 0$ and P is constant with respect to n.

(b)

$$\begin{aligned}\frac{\partial P}{\partial i} &= -nV(1+i)^{-n-1} + rV\left(\frac{n(1+i)^{-n-1}i - (1-(1+i)^{-n})}{i^2}\right)\\ &= \frac{rnV}{i(1+i)^{n+1}} - \frac{nV}{(1+i)^{n+1}} - \frac{rV}{i}\left(\frac{1-(1+i)^{-n}}{i}\right)\end{aligned}$$

Multiplying through by $\frac{i(1+i)^{n+1}}{V}$, we get:

$$\begin{aligned}\frac{i(1+i)^{n+1}}{V}\frac{\partial P}{\partial i} &= n(r-i) - r(1+i)\left(\frac{(1+i)^n - 1}{i}\right)\\ &= n(r-i) - r(1+i)s_{\overline{n}|i}\\ &= n(r-i) - r(1+i)(1 + (1+i) + (1+i)^2 + \cdots + (1+i)^{n-1})\\ &= n(r-i) - r((1+i) + (1+i)^2 + (1+i)^3 + \cdots + (1+i)^n)^n)\end{aligned}$$

And by our assumption that (for $n \geq 2$) $(1+i)^n > 1+ni$, hence $-(1+i)^n < -(1+ni)$. (Note: those who know the binomial theorem may wish to prove this. Also note that if $n = 0, 1$ the $(1+i)^n = 1 + ni$.)

$$\begin{aligned}&< n(r-i) - r((1+i) + (1+2i) + (1+3i) + \cdots + (1+ni))\\ &= n(r-i) - r\left(n + i\left(\frac{n(n+1)}{2}\right)\right)\\ &= -ni - \frac{rin(n+1)}{2}\\ &< 0\end{aligned}$$

So $\frac{i(1+i)^{n+1}}{V}\frac{\partial P}{\partial i} < 0$, and since $\frac{i(1+i)^{n+1}}{V} > 0$, then $\frac{\partial P}{\partial i} < 0$, and thus price decreases as the yield increases.

10.

1) $\frac{\partial}{\partial x}(xy^2z^3 + x^3y^2z) = \frac{\partial}{\partial x}(x + y + z)$

$$\begin{aligned}&\Rightarrow y^2z^3 + 3xy^2z^2\left(\frac{\partial z}{\partial x}\right) + 3x^2y^2z + x^3y^2\left(\frac{\partial z}{\partial x}\right) = 1 + \frac{\partial z}{\partial x}\\ &\Rightarrow \left(\frac{\partial z}{\partial x}\right)(3xy^2z^2 + x^3y^2 - 1) = 1 - y^2z^3 - 3x^2y^2z\\ &\Rightarrow \frac{\partial z}{\partial x} = \frac{1 - y^2z^3 - 3x^2y^2z}{3xy^2z^2 + x^3y^2 - 1}\end{aligned}$$

2) $\frac{\partial}{\partial y}(xy^2z^3 + x^3y^2z) = \frac{\partial}{\partial y}(x+y+z)$

$$\Rightarrow\ 2xyz^3 + 3xy^2z^2\left(\frac{\partial z}{\partial y}\right) + 2x^3yz + x^3y^2\left(\frac{\partial z}{\partial y}\right) = 1 + \frac{\partial z}{\partial y}$$
$$\Rightarrow\ \left(\frac{\partial z}{\partial y}\right)(3xy^2z^2 + x^3y^2 - 1) \ = \ 1 - 2xyz^3 - 2x^3yz$$
$$\Rightarrow\ \frac{\partial z}{\partial y} \ = \ \frac{1 - 2xyz^3 - 2x^3yz}{3xy^2z^2 + x^3y^2 - 1}$$

11.

i) $\frac{\partial}{\partial x}(z\ln(x^2+y^2)) \ = \ \frac{\partial}{\partial x}(1).$

$$\Rightarrow \frac{\partial z}{\partial x}\ln(x^2+y^2) + \frac{2xz}{x^2+y^2} \ = \ 0 \ \Rightarrow \ \frac{\partial z}{\partial x} \ = \ \frac{-2xz}{(x^2+y^2)\ln(x^2+y^2)}.$$

ii) By symmetry of x and y, $\frac{\partial z}{\partial y} \ = \ \frac{-2yz}{(x^2+y^2)\ln(x^2+y^2)}$. So

$$\begin{aligned} x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} \ &= \ \frac{-2x^2z}{(x^2+y^2)\ln(x^2+y^2)} - \frac{2y^2z}{(x^2+y^2)\ln(x^2+y^2)} \\ &= \ \frac{(x^2+y^2)(-2z)}{(x^2+y^2)\ln(x^2+y^2)} \\ &= \ -\frac{2z}{\ln(x^2+y^2)} \qquad \left(\text{But } \frac{1}{\ln(x^2+y^2)} = z, \text{ so}\right) \\ &= \ -2z^2. \end{aligned}$$

12.

(a) Calculating $\frac{\partial V}{\partial P}$:

$$\left[1 - \frac{2a}{V^3}\left(\frac{\partial V}{\partial P}\right)\right](V-b) + (P + \frac{a}{V^2})\left(\frac{\partial V}{\partial P}\right) = 0$$
$$\Rightarrow\ \left(P + \frac{a}{V^2}\right)\frac{\partial V}{\partial P} + \frac{2a}{V^3}(b-V)\left(\frac{\partial V}{\partial P}\right) \ = \ b - V$$
$$\Rightarrow\ \frac{\partial V}{\partial P} \ = \ \frac{b-V}{P + \frac{a}{V^2} + \frac{2a(b-V)}{V^3}} \ = \ \frac{b-V}{\frac{V^3P + aV + 2a(b-V)}{V^3}} \ = \ \frac{V^3(b-V)}{V^3P + aV + 2a(b-V)}$$

Also, before the change, $V = 25,600$ cm^3 and $P = 1$ atm., so

$$\left.\frac{\partial V}{\partial P}\right|_{\substack{V=25,600cm^3 \\ P=1\ atm.}} \ = \ \frac{(25,600)^3(42.7 - 25,600)}{(25,600)^3(1) + (3.59\times10^6)(25,600) + 2(3.59\times10^6)(42.7-25,600)}$$
$$\approx \ \frac{-4.2878\times10^{17}}{1.6686\times10^{13}} \ \approx \ -25,698\frac{cm^3}{atm}$$

Hence $\frac{\Delta V}{\Delta P} \approx \frac{\partial V}{\partial P}$, so

$$\begin{aligned} \Delta V &\approx \frac{\partial V}{\partial P}\Delta P \\ &= -25,698(+.1\,atm) \\ &\approx -2570\ cm^3 \end{aligned}$$

(b) Calculating $\frac{\partial V}{\partial T}$:

$$\begin{aligned} -\frac{2a}{V^3}\left(\frac{\partial V}{\partial T}\right)(V-b) + (P+\frac{a}{V^2})\left(\frac{\partial V}{\partial T}\right) &= 82.06 \\ \Rightarrow \frac{\partial V}{\partial T}(P+\frac{a}{V^2}+\frac{2a(b-V)}{V^3}) &= 82.06 \end{aligned}$$

$$\Rightarrow \frac{\partial V}{\partial T} = \frac{82.06V^3}{V^3P+Va+2a(b-V)} = \frac{82.06V^3}{V^3P-Va+2ab}$$

Again, $V = 25,600\,cm^3$ and $P = 1\,atm.$:

$$\begin{aligned} \left.\frac{\partial V}{\partial T}\right|_{\substack{V=25,600cm^3 \\ P=1\ atm.}} &= \frac{82.06(25,600)^3}{(25,600)^3(1)-(3.59\times 10^6)(25,600)+2(3.59\times 10^6)(42.7)} \\ &\approx \frac{1.3767\times 10^{15}}{1.6689\times 10^{13}} \approx 82.5\frac{cm^3}{{}^oK} \end{aligned}$$

Hence $\frac{\Delta V}{\Delta T} \approx \frac{\partial V}{\partial T}$, so

$$\begin{aligned} \Delta V &\approx \frac{\partial V}{\partial T}\Delta T \\ &= 82.5(+1^oK) \\ &= 82.5\ cm^3. \end{aligned}$$

13.

1) differentiating with respect to x:

$$3z^2\left(\frac{\partial z}{\partial x}\right) - x\frac{\partial z}{\partial x} - z = 0, \quad \text{so} \quad \frac{\partial z}{\partial x} = \frac{z}{3z^2-x}$$

2) differentiating with respect to y:

$$3z^2\left(\frac{\partial z}{\partial y}\right) - x\frac{\partial z}{\partial y} - 1 = 0, \quad \text{so} \quad \frac{\partial z}{\partial y} = \frac{1}{3z^2-x}$$

Differentiating (2) with respect to x and using (1):

$$\begin{aligned}\frac{\partial^2 z}{\partial x \partial y} &= \frac{\partial}{\partial x}\left(\frac{1}{3z^2 - x}\right) = \left(\frac{-1}{(3z^2-x)^2}\right)\left(6z\left(\frac{\partial z}{\partial x}\right) - 1\right) \\ &= \frac{1 - 6z\left[\frac{z}{(3z^2-x)}\right]}{(3z^2-x)^2} = -\frac{(3z^2+x)}{(3z^2-x)^3}\end{aligned}$$

14. (a)

$$\begin{aligned}f_x(x,y) &= \frac{(3x^2y - y^3)(x^2+y^2) - 2x(x^3y - xy^3)}{(x^2+y^2)^2} \\ &= \frac{x^4y + 4x^2y^3 - y^5}{(x^2+y^2)^2} \\ f_y(x,y) &= \frac{(x^3 - 3xy^2)(x^2+y^2) - 2y(x^3y - xy^3)}{(x^2+y^2)^2} \\ &= \frac{x^5 + 4x^3y^2 - xy^4}{(x^2+y^2)^2}\end{aligned}$$

(b)

$$\begin{aligned}f_x(0,0) &= \lim_{h\to 0} \frac{f(h,0) - f(0,0)}{h} \\ &= \lim_{h\to 0} \frac{\frac{h^3(0) - h(0)^3}{h^2+0^2} - 0}{h} \\ &= \lim_{h\to 0} \frac{\frac{0}{h^2} - 0}{h} = \lim_{h\to 0} \frac{0}{h} = \lim_{h\to 0} 0 = 0\end{aligned}$$

$$\begin{aligned}f_y(0,0) &= \lim_{h\to 0} \frac{f(0,h) - f(0,0)}{h} \\ &= \lim_{h\to 0} \frac{\frac{(0)^3h - 0(h)^3}{0^2+h^2} - 0}{h} \\ &= \lim_{h\to 0} \frac{\frac{0}{h^2} - 0}{h} = \lim_{h\to 0} \frac{0}{h} = \lim_{h\to 0} 0 = 0.\end{aligned}$$

(c)

$$\begin{aligned}f_{xy}(0,0) &= \frac{\partial}{\partial y}(f_x(x,y))\bigg|_{\substack{x=0\\y=0}} = \lim_{h\to 0} \frac{f_x(0,h) - f_x(0,0)}{h} \\ &= \lim_{h\to 0} \frac{\frac{(0)^4h + 4(0)^2h^3 - h^5}{(0^2+h^2)^2} - 0}{h} \\ &= \lim_{h\to 0} \frac{\frac{-h^5}{h^4}}{h} = \lim_{h\to 0} -1 = -1\end{aligned}$$

$$
\begin{aligned}
f_{yx}(0,0) &= \frac{\partial}{\partial x}(f_y(x,y))\Bigg|_{\substack{x=0\\y=0}} = \lim_{h\to 0}\frac{f_y(h,0)-f_y(0,0)}{h} \\
&= \lim_{h\to 0}\frac{\frac{h^5+4(h)^3(0)^2-h(0)^4}{(h^2+0^2)^2}-0}{h} \\
&= \lim_{h\to 0}\frac{\frac{h^5}{h^4}}{h} = \lim_{h\to 0} 1 = 1
\end{aligned}
$$

15. (a)

$$
\frac{\partial u}{\partial x} = \frac{\partial}{\partial x}(\ln(\sqrt{x^2+y^2})) = \frac{1}{\sqrt{x^2+y^2}}\left(\frac{2x}{2\sqrt{x^2+y^2}}\right) = \frac{x}{x^2+y^2},
$$

so

$$
\frac{\partial^2 u}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x}\right) = \frac{\partial}{\partial x}\left(\frac{x}{x^2+y^2}\right) = \frac{x^2+y^2-2x(x)}{(x^2+y^2)^2} = \frac{y^2-x^2}{(x^2+y^2)^2}.
$$

By symmetry, $\frac{\partial^2 u}{\partial y^2} = \frac{x^2-y^2}{(x^2+y^2)^2}$ and now $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = \frac{y^2-x^2}{(x^2+y^2)^2} + \frac{x^2-y^2}{(x^2+y^2)^2} = 0$ and LaPlace's equation is satisfied by $\ln(\sqrt{x^2+y^2})$

(b)

$$
\frac{\partial u}{\partial x} = \frac{\partial}{\partial x}(\sqrt{x^2+y^2}) = \frac{2x}{2\sqrt{x^2+y^2}} = \frac{x}{\sqrt{x^2+y^2}},
$$

$$
\begin{aligned}
\frac{\partial^2 u}{\partial x^2} &= \frac{\partial}{\partial x}\left(\frac{x}{\sqrt{x^2+y^2}}\right) = \frac{\sqrt{x^2+y^2}-\left(\frac{x}{\sqrt{x^2+y^2}}\right)x}{x^2+y^2} \\
&= \frac{x^2+y^2-x^2}{(x^2+y^2)^{\frac{3}{2}}} = \frac{y^2}{(x^2+y^2)^{\frac{3}{2}}}
\end{aligned}
$$

By symmetry, $\frac{\partial^2 u}{\partial y^2} = \frac{x^2}{(x^2+y^2)^{\frac{3}{2}}}$. And so

$$
\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = \frac{y^2}{(x^2+y^2)^{\frac{3}{2}}} + \frac{x^2}{(x^2+y^2)^{\frac{3}{2}}} = \frac{x^2+y^2}{(x^2+y^2)^{\frac{3}{2}}} = \frac{1}{\sqrt{x^2+y^2}}.
$$

So $\sqrt{x^2+y^2}$ does not satisfy LaPlace's equation.

16. Consider any $i, 1 \le i \le n$. Then by the Chain Rule (considering $x_j, j \ne i$, as constants),

$$
\frac{\partial u}{\partial x_i} = a_i e^{\sum_{i=1}^n a_i x_i} = a_i u \quad \text{and}
$$

$$
\begin{aligned}
\frac{\partial^2 u}{\partial x_i^2} &= \frac{\partial u}{\partial x_i}\left(\frac{\partial u}{\partial x_i}\right) = \frac{\partial}{\partial x_i}(a_i u) \\
&= a_i\left(\frac{\partial u}{\partial x_i}\right) = a_i(a_i u) = a_i^2 u
\end{aligned}
$$

So $\sum_{i=1}^{n} \frac{\partial^2 u}{\partial x_i^2} = \sum_{i=1}^{n} a_i^2 u = u(\sum_{i=1}^{n} a_i^2)$. But it is given that $\sum_{i=1}^{n} a_i^2 = 1$, and the answer is u.

17.

$$z = xe^y + ye^x$$

$\frac{\partial}{\partial x}$ $\quad\quad$ $\frac{\partial}{\partial y}$

$\frac{\partial z}{\partial x} = e^y + ye^x$ $\quad\quad$ $\frac{\partial z}{\partial y} = xe^y + e^x$

$\frac{\partial}{\partial x}$ $\quad\quad$ $\frac{\partial}{\partial x}$ $\quad\quad$ $\frac{\partial}{\partial y}$

$\frac{\partial^2 z}{\partial x^2} = ye^x$ $\quad\quad$ $\frac{\partial^2 z}{\partial x \partial y} = e^y + e^x$ $\quad\quad$ $\frac{\partial^2 z}{\partial y^2} = xe^y$

$\frac{\partial}{\partial x}$ $\quad\quad$ $\frac{\partial}{\partial x}$ $\quad\quad$ $\frac{\partial}{\partial x}$ $\quad\quad$ $\frac{\partial}{\partial y}$

$\frac{\partial^3 z}{\partial x^3} = ye^x$ $\quad\quad$ $\frac{\partial^3 z}{\partial x^2 \partial y} = e^x$ $\quad\quad$ $\frac{\partial^3 z}{\partial y^2 \partial x} = e^y$ $\quad\quad$ $\frac{\partial^3 z}{\partial y^3} = xe^y$

So now,

$$\frac{\partial^3 z}{\partial x^3} + \frac{\partial^3 z}{\partial y^3} = ye^x + xe^y = x\frac{\partial^3 z}{\partial x \partial y^2} + y\frac{\partial^3 z}{\partial x^2 \partial y}.$$

18.

Solving for r and s in terms of x and y, we get: $r = \frac{(2x+y)}{5}$ and $s = \frac{(2y-x)}{5}$. So $\frac{\partial r}{\partial x} = \frac{2}{5}$ and $\frac{\partial s}{\partial x} = -\frac{1}{5}$

Also, $\frac{\partial r}{\partial y} = \frac{1}{5}$ and $\frac{\partial s}{\partial y} = \frac{2}{5}$

Now, $\frac{\partial U}{\partial x} = \frac{\partial U}{\partial r}\frac{\partial r}{\partial x} + \frac{\partial U}{\partial s}\frac{\partial s}{\partial x} = \frac{2}{5}\frac{\partial U}{\partial r} - \frac{1}{5}\frac{\partial U}{\partial s}$. And

$$\begin{aligned}
\frac{\partial^2 U}{\partial y \partial x} &= \frac{\partial}{\partial y}\left(\frac{\partial U}{\partial x}\right) = \frac{\partial U_x}{\partial y} = \frac{\partial U_x}{\partial r}\frac{\partial r}{\partial y} + \frac{\partial U_x}{\partial s}\frac{\partial s}{\partial y} \\
&= \left[\frac{\partial}{\partial r}\left(\frac{2}{5}\frac{\partial U}{\partial r} - \frac{1}{5}\frac{\partial U}{\partial s}\right)\right]\left(\frac{1}{5}\right) + \left[\frac{\partial}{\partial s}\left(\frac{2}{5}\frac{\partial U}{\partial r} - \frac{1}{5}\frac{\partial U}{\partial s}\right)\right]\left(\frac{2}{5}\right) \\
&= \left(\frac{2}{5}\frac{\partial^2 U}{\partial r^2} - \frac{1}{5}\frac{\partial^2 U}{\partial r \partial s}\right)\left(\frac{1}{5}\right) + \left(\frac{2}{5}\frac{\partial U^2}{\partial s \partial r} - \frac{1}{5}\frac{\partial^2 U}{\partial s^2}\right)\left(\frac{2}{5}\right) \\
&= \frac{1}{25}\left(2\frac{\partial^2 U}{\partial r^2} - \frac{\partial^2 U}{\partial r \partial s} + 4\frac{\partial^2 U}{\partial s \partial r} - 2\frac{\partial^2 U}{\partial s^2}\right)
\end{aligned}$$

And assuming that $\frac{\partial^2 U}{\partial r \partial s} = \frac{\partial^2 U}{\partial s \partial r}$ (i.e. U has continuous second partial derivatives).

Then

$$= \frac{1}{25}\left(2\frac{\partial^2 U}{\partial r^2} + 3\frac{\partial^2 U}{\partial s \partial r} - 2\frac{\partial^2 U}{\partial s^2}\right).$$

(b) As in part (a),

$$\frac{\partial U}{\partial y \partial x} = \frac{1}{25}\left(2\frac{\partial^2 U}{\partial r^2} + 3\frac{\partial^2 U}{\partial s \partial r} - 2\frac{\partial^2 U}{\partial s^2}\right).$$

We find $\frac{\partial^2 U}{\partial r^2}$, $\frac{\partial^2 U}{\partial s \partial r}$ and $\frac{\partial^2 U}{\partial s^2}$

$$U = r^2 e^{rs}$$

$\frac{\partial}{\partial r}$ $\frac{\partial}{\partial s}$

So $\frac{\partial U}{\partial r} = 2re^{rs} + sr^2 e^{rs}$ And $\frac{\partial U}{\partial s} = r^3 e^{rs}$

$\frac{\partial}{\partial r}$ $\frac{\partial}{\partial s}$ $\frac{\partial}{\partial s}$

$$\begin{aligned}\frac{\partial^2 U}{\partial r^2} &= 2e^{rs} + 2rse^{rs} + 2rse^{rs} \\ &\quad + r^2 s^2 e^{rs} \\ &= (rs)^2 e^{rs} + 4rse^{rs} + 2e^{rs} \\ &= e^{rs}((rs)^2 + 4rs + 2)\end{aligned}$$

$$\begin{aligned}\frac{\partial^2 U}{\partial s \partial r} &= 2r^2 e^{rs} + r^2 e^{rs} + sr^3 e^{rs} \\ &= 3r^2 e^{rs} + sr^3 e^{rs} \\ &= e^{rs}(3r^2 + sr^3)\end{aligned}$$

$$\frac{\partial^2 U}{\partial s^2} = r^4 e^{rs}$$

[Note that all second partial derivatives are continuous, and hence $\frac{\partial^2 U}{\partial r \partial s} \equiv \frac{\partial^2 U}{\partial s \partial r}$

Now,

$$\begin{aligned}\frac{\partial^2 U}{\partial y \partial x} &= \frac{1}{25}(2(e^{rs}((rs)^2 + 4rs + 2)) + 3(e^{rs}(3r^2 + sr^3)) - 2(r^4 e^{rs})) \\ &= \frac{e^{rs}}{25}(-2r^4 + 3r^3 s + 2r^2 s^2 + 9r^2 + 8rs + 4)\end{aligned}$$

19.

$$\begin{aligned}\frac{\partial z}{\partial x} &= [f'(x^2 - y^2)]2x \\ \frac{\partial z}{\partial y} &= [f'(x^2 - y^2)](-2y) + 1\end{aligned}$$

Therefore

$$\begin{aligned}y\frac{\partial z}{\partial x} + x\frac{\partial z}{\partial y} &= 2xy[f'(x^2 - y^2)] - 2xy[f'(x^2 - y^2)] + x \\ &= x\end{aligned}$$

20.

Let $u = x + at$. Let $v = x - at$. Then

$$\begin{aligned}
\frac{\partial Y}{\partial t} &= \frac{\partial Y}{\partial u}\frac{\partial u}{\partial t} + \frac{\partial Y}{\partial v}\frac{\partial v}{\partial t} = af'(u) - ag'(v) \\
\frac{\partial Y}{\partial x} &= \frac{\partial Y}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial Y}{\partial v}\frac{\partial v}{\partial x} = f'(u) + g'(v)
\end{aligned}$$

Also,

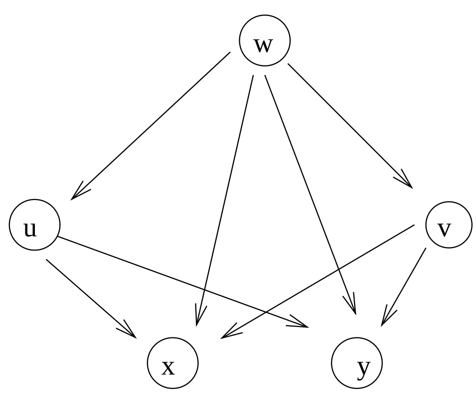
$$\begin{aligned}
\frac{\partial^2 Y}{\partial t^2} &= \frac{\partial}{\partial t}\left(\frac{\partial Y}{\partial t}\right) = \frac{\partial}{\partial t}(Y_t) = \frac{\partial Y_t}{\partial u}\frac{\partial u}{\partial t} + \frac{\partial Y_t}{\partial v}\frac{\partial v}{\partial t} \\
&= \left[\frac{\partial}{\partial u}(af'(u) - ag'(v))\right](a) + \left[\frac{\partial}{\partial v}(af'(u) - ag'(v))\right](-a) \\
&= a^2 f''(u) + a^2 g''(v).
\end{aligned}$$

And

$$\begin{aligned}
\frac{\partial^2 Y}{\partial x^2} &= \frac{\partial Y_x}{\partial x} = \frac{\partial Y_x}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial Y_x}{\partial v}\frac{\partial v}{\partial x} \\
&= \frac{\partial}{\partial u}(f'(u) + g'(v)) + \frac{\partial}{\partial v}(f'(u) + g'(v)) \\
&= f''(u) + g''(v)
\end{aligned}$$

So $\frac{\partial^2 Y}{\partial t^2} = a^2(f''(u) + g''(v)) = a^2\left(\frac{\partial^2 Y}{\partial x^2}\right)$.

21. For these questions, consider the following:



In each case we consider the eight partial derivatives in the above illustration.

(a)

$$\begin{aligned}
&1)\quad \frac{\partial f}{\partial u} = \frac{\partial}{\partial u}\sqrt{uvxy} = \frac{\sqrt{vxy}}{2\sqrt{u}} = \frac{1}{2}\frac{(x+y)^{\frac{1}{4}}}{(x-y)^{\frac{1}{4}}}(xy)^{\frac{1}{2}} \\
&2)\quad \frac{\partial f}{\partial v} = \frac{\partial}{\partial v}\sqrt{uvxy} = \frac{\sqrt{uxy}}{2\sqrt{v}} = \frac{1}{2}\frac{(x-y)^{\frac{1}{4}}}{(x+y)^{\frac{1}{4}}}(xy)^{\frac{1}{2}} \\
&3)\quad \frac{\partial f}{\partial x} = \frac{\partial}{\partial x}\sqrt{uvxy} = \frac{\sqrt{uvy}}{2\sqrt{x}} = \frac{1}{2}(x^2 - y^2)^{\frac{1}{4}}\left(\frac{y}{x}\right)^{\frac{1}{2}}
\end{aligned}$$

4) $\frac{\partial f}{\partial y} = \frac{\partial}{\partial y}\sqrt{uvxy} = \frac{\sqrt{uvx}}{2\sqrt{y}} = \frac{1}{2}(x^2-y^2)^{\frac{1}{4}}(\frac{x}{y})^{\frac{1}{2}}$

5) $\frac{\partial u}{\partial x} = \frac{\partial}{\partial x}\sqrt{x-y} = \frac{1}{2\sqrt{x-y}}$

6) $\frac{\partial u}{\partial y} = \frac{\partial}{\partial y}\sqrt{x-y} = \frac{-1}{2\sqrt{x-y}}$

7) $\frac{\partial v}{\partial x} = \frac{\partial}{\partial x}\sqrt{x+y} = \frac{1}{2\sqrt{x+y}}$

8) $\frac{\partial v}{\partial y} = \frac{\partial}{\partial y}\sqrt{x+y} = \frac{1}{2\sqrt{x+y}}$

So

$$\begin{aligned}
\frac{\partial w}{\partial x} &= \frac{\partial f}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial f}{\partial v}\frac{\partial v}{\partial x} + \frac{\partial f}{\partial x} \\
&= \frac{1}{2}\frac{(x+y)^{\frac{1}{4}}}{(x-y)^{\frac{1}{4}}}(xy)^{\frac{1}{2}}\left(\frac{1}{2}\frac{1}{(x-y)^{\frac{1}{2}}}\right) + \frac{1}{2}\frac{(x-y)^{\frac{1}{4}}}{(x+y)^{\frac{1}{4}}}(xy)^{\frac{1}{2}}\left(\frac{1}{2}\frac{1}{(x+y)^{\frac{1}{2}}}\right) \\
&\quad + \frac{1}{2}(x^2-y^2)^{\frac{1}{4}}(\frac{y}{x})^{\frac{1}{2}} \\
&= \frac{1}{4}\frac{(x+y)^{\frac{1}{4}}(xy)^{\frac{1}{2}}}{(x-y)^{\frac{3}{4}}} + \frac{1}{4}\frac{(x-y)^{\frac{1}{4}}(xy)^{\frac{1}{2}}}{(x+y)^{\frac{3}{4}}} + \frac{1}{2}(x^2-y^2)^{\frac{1}{4}}(\frac{y}{x})^{\frac{1}{2}}
\end{aligned}$$

Also,

$$\begin{aligned}
\frac{\partial w}{\partial y} &= \frac{\partial f}{\partial u}\frac{\partial u}{\partial y} + \frac{\partial f}{\partial v}\frac{\partial v}{\partial y} + \frac{\partial f}{\partial y} \\
&= \frac{1}{2}\frac{(x+y)^{\frac{1}{4}}(xy)^{\frac{1}{2}}}{(x-y)^{\frac{1}{4}}}\left(\frac{1}{2}\frac{-1}{(x-y)^{\frac{1}{2}}}\right) + \frac{1}{2}\frac{(x-y)^{\frac{1}{4}}(xy)^{\frac{1}{2}}}{(x+y)^{\frac{1}{4}}}\left(\frac{1}{2}\frac{1}{(x+y)^{\frac{1}{2}}}\right) \\
&\quad + \frac{1}{2}(x^2-y^2)^{\frac{1}{4}}(\frac{x}{y})^{\frac{1}{2}} \\
&= -\frac{1}{4}\frac{(x+y)^{\frac{1}{4}}(xy)^{\frac{1}{2}}}{(x-y)^{\frac{3}{4}}} + \frac{1}{4}\frac{(x-y)^{\frac{1}{4}}(xy)^{\frac{1}{2}}}{(x+y)^{\frac{3}{4}}} + \frac{1}{2}(x^2-y^2)^{\frac{1}{4}}(\frac{x}{y})^{\frac{1}{2}}
\end{aligned}$$

(b)

1) $\frac{\partial f}{\partial u} = v = \frac{y}{x^2+y^2}$

2) $\frac{\partial f}{\partial v} = u = \frac{x}{x^2+y^2}$

3) $\frac{\partial f}{\partial x} = -y$

4) $\frac{\partial u}{\partial x} = \frac{(x^2+y^2)-2x(x)}{(x^2+y^2)^2} = \frac{y^2-x^2}{(x^2+y^2)^2}$

5) $\frac{\partial v}{\partial x} = \frac{-2xy}{(x^2+y^2)^2}$

So

$$\begin{aligned}
\frac{\partial w}{\partial x} &= \frac{\partial f}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial f}{\partial v}\frac{\partial v}{\partial x} + \frac{\partial f}{\partial x} \\
&= \frac{y}{x^2+y^2}\left(\frac{y^2-x^2}{(x^2+y^2)^2}\right) + \frac{x}{x^2+y^2}\left(\frac{-2xy}{(x^2+y^2)^2}\right) - y \\
&= \frac{y^3-x^2y-2x^2y}{(x^2+y^2)^3} - y = \frac{y^3-3x^2y}{(x^2+y^2)^3} - y
\end{aligned}$$

And by symmetry it follows that

$$\frac{\partial w}{\partial y} = \frac{x^3 - 3y^2x}{(x^2+y^2)^3} - x.$$

22. We need $f_x(x,y) = f_y(x,y) = 0$.

i) $f_x(x,y) = 0 \Rightarrow 4x^3 - 8y = 0 \Rightarrow x^3 = 2y$

ii) $f_y(x,y) = 0 \Rightarrow -8x + 4y = 0 \Rightarrow y = 2x$

Combining these, we get

iii) $x^3 = 2(2x) = 4x = \Rightarrow x(x-2)(x+2) = 0 \Rightarrow x = -2, 0, 2.$

So we obtain the critical points $(-2,-4), (0,0)$, and $(2,4)$. To classify these critical points, we need to find $D(x_0,y_0) = f_{xx}(x_0,y_0)f_{yy}(x_0,y_0) - [f_{xy}(x_0,y_0)]^2$ for each (x_0,y_0) above:

$$f_{xx}(x,y) = 12x^2; \quad f_{yy}(x,y) = 4; \quad f_{xy}(x,y) = -8.$$

(a) $(x_0,y_0) = (-2,-4)$.

Here $D(-2,-4) = 12(-2)^2(4) - [-8]^2 = 192 - 64 = 128$. So $D(-2,-4) > 0$. Also, $f_{xx}(-2,-4) = 12(-2)^2 = 48 > 0$, and hence (-2,-4) is a relative minimum for $f(x,y)$.

(b) $(x_0,y_0) = (0,0)$

Here $D(0,0) = 12(0)^2(4) - [-8]^2 = 0 - 64 = -64$. So $D(0,0) < 0$, and $f(x,y)$ has neither a relative maximum nor minimum at $(0,0)$.

(c) $(x_0,y_0) = (2,4)$.

Here $D(2,4) = 12(2)^2(4) - [-8]^2 = 196 - 64 = 128$. So $D(2,4) > 0$ and $f_{xx}(2,4) = 12(2)^2 = 48 > 0$, so f has a relative minimum at $(2,4)$.

23. The girth is $2h + 2w$. So the imposed constraint is $\ell + 2h + 2w \leq 108$.

When looking for the maximum possible volume, it is obvious that this will occur when strict equality holds in the above equation. So we have:

$$\ell + 2h + 2w = 108, \quad \text{or} \quad \ell = 108 - 2h - 2w.$$

We must maximize the volume $V = \ell hw$. So we maximize $V = (108 - 2h - 2w)hw = 108hw - 2h^2w - 2hw^2$. We need to find the following partial derivatives:

(1) $\frac{\partial V}{\partial w} = 108h - 2h^2 - 4hw$

(2) $\frac{\partial^2 V}{\partial w^2} = \frac{\partial}{\partial w}\left(\frac{\partial V}{\partial w}\right) - 4h$

(3) $\frac{\partial^2 V}{\partial h \partial w} = \frac{\partial}{\partial h}\left(\frac{\partial V}{\partial w}\right) = 108 - 4h - 4hw$

(4) $\frac{\partial V}{\partial h} = 108w - 4hw - 2w^2$

(5) $\frac{\partial^2 V}{\partial h^2} \;=\; -4w.$

V

$\frac{\partial}{\partial w}$ $\qquad$ $\frac{\partial}{\partial h}$

(1) $\frac{\partial V}{\partial w}$ $\qquad$ (4) $\frac{\partial V}{\partial h}$

$\frac{\partial}{\partial w}$ $\qquad$ $\frac{\partial}{\partial h}$ $\qquad$ $\frac{\partial}{\partial h}$

(2) $\frac{\partial^2 V}{\partial w^2}$ $\qquad$ (3) $\frac{\partial^2 V}{\partial h \partial w}$ $\qquad$ (5) $\frac{\partial^2 V}{\partial h^2}$

Now, we set $\frac{\partial V}{\partial w}$ and $\frac{\partial V}{h}$ to zero.

1)

$$\begin{aligned}
\frac{\partial V}{\partial w} &= 108h - 2h^2 - 4hw \;=\; 0 \\
\Rightarrow\; h(108 - 2h - 4w) &= 0 \\
\Rightarrow\; 108 - 2h - 4w &= 0 \quad \text{(obviously } h \text{ is not zero)} \\
\Rightarrow\; h + 2w &= 54.
\end{aligned}$$

2) By symmetry, when we set $\frac{\partial V}{\partial h}$ to zero, we get $w + 2h = 54$, or $2h + w = 54$.

So we obtain $h + 2w = 54$ and $2h + w = 54$. We solve and get $h = w = 18$. To ensure that this gives a maximum, we use the second derivative test:

$$\begin{aligned}
\frac{\partial^2 V}{\partial h^2}\left(\frac{\partial^2 V}{\partial w^2}\right) - \left(\frac{\partial^2 V}{\partial h \partial w}\right)^2 &= (-4w)(-4h) - (108 - 4w - 4h)^2 (\text{at } h = w = 18) \\
&= 16(18)^2 - (108 - 8(18))^2 \\
&= 16(18)^2 - [-2(18)]^2 = 12[18]^2 > 0
\end{aligned}$$

Also, $\frac{\partial^2 V}{\partial h^2} = -4w = -4(18) < 0$ and so the volume is indeed maximized at $h = w = 18$ in. The length here is $\ell = 108 - 2h - 2w = 36$ in., and the volume is:

$$V = \ell hw = 36(18)^2 = 11,664 \text{ in}^3 = 6\frac{3}{4}\, ft^3.$$

24. The revenue function is $R = P_1Q_1 + P_2Q_2$. So the profit function is $\pi = R - C = P_1Q_1 + P_2Q_2 - 2Q_1^2 - Q_1Q_2 - 2Q_2^2$. We find $\frac{\partial \pi}{\partial Q_1}$ and $\frac{\partial \pi}{\partial Q_2}$ and set them to zero:

1) $\frac{\partial \pi}{\partial Q_1} = 0 \;\Rightarrow\; P_1 - 4Q_1 - Q_2 = 0 \;\Rightarrow\; 4Q_1 + Q_2 = P_1$

2) $\frac{\partial\pi}{\partial Q_2} = 0 \;\Rightarrow\; P_2 - Q_1 - 4Q_2 = 0 \;\Rightarrow\; Q_1 + 4Q_2 = P_2$

We obtain $Q_1 = \frac{4P_1 - P_2}{15}$ and $Q_2 = \frac{4P_2 - P_1}{15}$. To ensure that this is a maximum, we use the second-derivative test:

$$\frac{\partial\pi}{\partial Q_1^2} = -4; \;\; \frac{\partial\pi}{\partial Q_1 \partial Q_2} \;\equiv\; -1; \;\; \frac{\partial\pi}{\partial Q_2^2} \equiv -4,$$

so $\frac{\partial\pi}{\partial Q_1^2}\frac{\partial\pi}{\partial Q_2^2} - \left(\frac{\partial\pi}{\partial Q_1 \partial Q_2}\right)^2 \;\equiv\; 15 > 0$ and $\frac{\partial\pi}{\partial Q_1^2} < 0$, and we do get a relative (in fact global) maximum at $(Q_1, Q_2) = \left(\frac{4P_1 - P_2}{15}, \frac{4P_2 - P_1}{15}\right)$.

25. The revenue function is

$$\begin{aligned} R = &\; P_1 Q_1 + P_2 Q_2 = (55 - Q_1 - Q_2)Q_1 + (70 - Q_1 - 2Q_2)Q_2 \\ = &\; 55Q_1 + 70Q_2 - Q_1^2 - 2Q_1Q_2 - 2Q_2^2. \end{aligned}$$

So the profit function π is $\pi = R - C = 55Q_1 + 70Q_2 - 2Q_1^2 - 3Q_1Q_2 - 3Q_2^2$. Again, we find $\frac{\partial\pi}{\partial Q_1}$ and $\frac{\partial\pi}{\partial Q_2}$ and set them to zero:

1) $\frac{\partial\pi}{\partial Q_1} \;=\; 0 \;\Rightarrow\; 55 - 4Q_1 - 3Q_2 = 0 \;\Rightarrow\; 4Q_1 + 3Q_2 = 55.$

2) $\frac{\partial\pi}{\partial Q_2} \;=\; 0 \;\Rightarrow\; 70 - 3Q_1 - 6Q_2 = 0 \;\Rightarrow\; 3Q_1 + 6Q_2 = 70.$

Using Linear Algebra, we get: $Q_1 = 8$ and $Q_2 = 7\frac{2}{3}$ Here, $P_1 = 55 - Q_1 - Q_2 = 55 - 8 - 7\frac{2}{3} = 39\frac{1}{3}$ and $P_2 = 70 - Q_1 - 2Q_2 = 70 - 8 - 2(7\frac{2}{3}) = 46\frac{2}{3}$

We must ensure that this is indeed a maximum for the profit function π. So we use the Second Derivative Test:

$$\frac{\partial^2\pi}{\partial Q_1^2} \;\equiv\; -4; \;\; \frac{\partial^2\pi}{\partial Q_1 \partial Q_2} \;\equiv\; -3; \;\; \frac{\partial^2\pi}{\partial Q_2^2} \;\equiv\; -6,$$

so $\frac{\partial^2\pi}{\partial Q_1^2}\frac{\partial^2\pi}{\partial^2 Q_2^2} - \left(\frac{\partial\pi}{\partial Q_1 \partial Q_2}\right)^2 \;\equiv\; 15 > 0$ and $\frac{\partial^2\pi}{\partial Q_1^2} < 0$ and this is a maximum.

26. Using Lagrange Multipliers:

$$\begin{aligned} F(x, y, z) &= xyz - \lambda(x^2 + y^2 + z^2 - 3) \\ \frac{\partial F}{\partial x} &= yz - \lambda 2x \\ \frac{\partial F}{\partial y} &= xz - \lambda 2y \\ \frac{\partial F}{\partial z} &= xy - \lambda 2z \end{aligned}$$

Setting these to zero, we get $\lambda = \frac{yz}{2x} = \frac{xz}{2y} = \frac{xy}{2z}(x, y, z \neq 0)$. $\quad (A)$

(Note if one of x, y, z is 0, eg. $x = 0$, then one of y or z must be zero also since $yz - \lambda 2x = 0$, eg. $y = 0$. But since this now forces $z \neq 0$ since $x^2 + y^2 + z^2 = 3$, then since $xy - \lambda 2z = 0$, we must have $\lambda = 0$. But this situation does not give rise to global maxima or minima, as we shall see)

From (A), $x^2 = y^2 = z^2$ (for example, $2x^2z = 2y^2z \Rightarrow x^2 = y^2$)

There are two cases:

1) None or two of x, y, z are negative: Here, xyz is positive and is $|x|^3$, where

$$x^2 + y^2 + z^2 = 3 \Rightarrow 3x^2 = 3 \Rightarrow x^2 = 1 \Rightarrow |x| = 1 \Rightarrow xyz = |x|^3 = 1.$$

2) One or three of x, y, z are negative: here, xyz is negative and is $-|x|^3$, where as in part (1), $|x| = 1$. Now $xyz = -|x|^3 = -1$. Now getting back to our parenthetical note, if, for example, $\lambda = x = y = 0$ and $z^2 = 3$, then obviously $xyz = 0$ is neither a global maximum or minimum.

27. We need to minimize $\sqrt{x^2 + y^2}$, or equivalently we minimize $(x^2 + y^2)$. According to the method of LaGrange multipliers, we consider:

$$F(x, y, \lambda) \;=\; x^2 + y^2 - \lambda(x^2 + 8xy + 7y^2 - 225).$$

Then $F_x(x, y, \lambda) = 2x - 2\lambda x - 8\lambda y = 0$, or

$$(\lambda - 1)x + 4\lambda y \;=\; 0 \qquad (1)$$

Also, $F_y(x, y, \lambda) \;=\; 2y - 8\lambda x - 14\lambda y = 0$, or

$$4\lambda x + (7\lambda - 1)y \;=\; 0. \qquad (2)$$

Since $(x, y) \neq (0, 0)$ (note that the hyperbola does not go through the origin) then, using Linear Algebra to solve for λ:

$$\begin{aligned}
\begin{vmatrix} (\lambda - 1) & 4\lambda \\ 4\lambda & (7\lambda - 1) \end{vmatrix} \;=\; 0 &\Rightarrow (\lambda - 1)(7\lambda - 1) - 16\lambda^2 = 0 \\
&\Rightarrow 7\lambda^2 - 7\lambda - \lambda + 1 - 16\lambda^2 = 0 \\
&\Rightarrow 9\lambda^2 + 8\lambda - 1 = 0 \\
&\Rightarrow \lambda = -1, \frac{1}{9}
\end{aligned}$$

A) If $\lambda = -1$, then $-2x - 4y = 0$, or $x = -2y$ and substitution in $x^2 + 8xy + 7y^2 = 225$ yields $-5y^2 = 225$, for which no real solution exists.

B) If $\lambda = \frac{1}{9}$, then $-\frac{8}{9}x + \frac{4}{9}y = 0$, or $y = 2x$ and substitution in $x^2 + 8xy + 7y^2 = 225$ yields $45x^2 = 225$. Then $x^2 = 5$, $y^2 = 4x^2 = 20$, and so $x^2 + y^2 = 25$. Thus the required shortest distance is $\sqrt{25} = 5$.

28.

1) We form $F(x, y, \lambda) = \frac{1}{x} + \frac{1}{y} - \lambda(\frac{1}{x^2} + \frac{1}{y^2} - 1)$ and obtain

$$F_x(x, y, \lambda) = -\frac{1}{x^2} + \frac{2\lambda}{x^3} = 0 \Rightarrow \frac{2\lambda}{x^3} = +\frac{1}{x^2} \Rightarrow 2\lambda = x,\ x \neq 0.$$

Also $F_y(x, y, \lambda) = -\frac{1}{y^2} + \frac{2\lambda}{y^3} = 0 \Rightarrow 2\lambda = y,\ y \neq 0$. Hence, $x = y = 2\lambda$, and since $\frac{1}{x^2} + \frac{1}{y^2} = 1$, $\frac{2}{x^2} = 1 \Rightarrow x = \pm\sqrt{2}$. So the critical points are $(\sqrt{2}, \sqrt{2})$ and $(-\sqrt{2}, -\sqrt{2})$.

2) We form $F(x, y, z, \lambda) = x + y + z - \lambda(\frac{1}{x} + \frac{1}{y} + \frac{1}{z} - 1)$. We obtain $F_x(x, y, z, \lambda) = 1 + \frac{\lambda}{x^2} = 0 \Rightarrow x^2 = -\lambda$. Similarly, when calculating $F_y(x, y, z, \lambda)$ and $F_z(x, y, z, \lambda)$, we obtain (by symmetry):

$$y^2 = -\lambda \quad \text{and} \quad z^2 = -\lambda.$$

Thus $x^2 = y^2 = z^2 (= -\lambda)$. Since $\frac{1}{x} + \frac{1}{y} + \frac{1}{z} = 1$, we get 2 cases:

1) $x = y = z$. Here $\frac{1}{x} + \frac{1}{y} + \frac{1}{z} = \frac{3}{x} = 1 \Rightarrow x = 3$, and the solution is $(x, y, z) = (3, 3, 3)$.
2) Two of x, y, z have the same sign and the other differs. Without loss of generality, let $x = y = -z$. Here, $\frac{1}{x} + \frac{1}{y} + \frac{1}{z} = \frac{1}{x} + \frac{1}{x} - \frac{1}{x} = \frac{1}{x} = 1$, so $x = 1$ (and $y = 1$ and $z = -1$).

Similarly, when $y = z = -x$, $y = z = 1$ and $x = -1$, and when $x = z = -y$, $x = z = 1$ and $y = -1$.

So we obtain the three solutions $(x, y, z) = (1, 1, -1), (1, -1, 1)$ and $(-1, 1, 1)$.

29. We wish to minimize $C(x, y) = mx + ny$ given the constraint $f(x, y) = kx^\alpha y^{1-\alpha} - Q = 0$ (Note that $x, y \neq 0$ since $Q > 0$, so that at any time division by a power of x, y is valid).

Using the method of Lagrange multipliers, we wish to find the critical points of

$$f(x, y, \lambda) = mx + ny - \lambda(kx^\alpha y^{1-\alpha} - Q).$$

Taking partial derivatives,

$$\begin{aligned} f_x &= m - \lambda k \alpha \left(\frac{y}{x}\right)^{1-\alpha} \\ f_y &= n - \lambda k (1 - \alpha) \left(\frac{x}{y}\right)^{\alpha} \\ f_\lambda &= -kx^\alpha y^{1-\alpha} + Q \end{aligned}$$

Setting $f_y = 0$ we have $n \ = \ \lambda k(1-\alpha)\left(\frac{x}{y}\right)^{\alpha} \ = \ \lambda k(1-\alpha)(\frac{y}{x})^{-\alpha}$.
Setting $f_x = 0$ we have $m = \lambda k\alpha(\frac{y}{x})^{1-\alpha}$
Dividing

$$\frac{n}{m} \ = \ \frac{1-\alpha}{\alpha}\left(\frac{x}{y}\right) \qquad \text{equation (1).}$$

Setting $f_\lambda = 0$, we have $kx^\alpha y^{1-\alpha} = Q$ so

$$\begin{array}{ll}
x^\alpha = \frac{Q}{k} y^{\alpha-1} & y^{1-\alpha} = \frac{Q}{k} x^{-\alpha} \\
x = \left(\frac{Q}{k}\right)^{\frac{1}{\alpha}} y^{\frac{\alpha-1}{\alpha}} & y = \left(\frac{Q}{k}\right)^{\frac{1}{1-\alpha}} x^{\frac{-\alpha}{1-\alpha}} \\
\frac{x}{y} = \left(\frac{Q}{k}\right)^{\frac{1}{\alpha}} y^{\frac{-1}{\alpha}} & \frac{y}{x} = \left(\frac{Q}{k}\right)^{\frac{1}{1-\alpha}} x^{\frac{-1}{1-\alpha}} \\
\text{equation (2)} & \text{equation (3)}
\end{array}$$

Combining equation (1) with equation (2) and combining equation (1) with equation (3) gives

$$\begin{array}{ll}
\frac{n}{m} = \frac{1-\alpha}{\alpha}\left(\frac{Q}{k}\right)^{\frac{1}{\alpha}} y^{-\frac{1}{\alpha}} & \frac{n}{m} = \frac{1-\alpha}{\alpha}\left(\frac{Q}{k}\right)^{-\frac{1}{1-\alpha}} x^{\frac{1}{1-\alpha}} \\
y^{\frac{1}{\alpha}} = \frac{m}{n}\cdot\frac{1-\alpha}{\alpha}\left(\frac{Q}{k}\right)^{\frac{1}{\alpha}} & x^{-\frac{1}{1-\alpha}} = \frac{m}{n}\cdot\frac{1-\alpha}{\alpha}\left(\frac{Q}{k}\right)^{-\frac{1}{1-\alpha}} \\
y = \left[\frac{m}{n}\cdot\frac{1-\alpha}{\alpha}\right]^{\alpha}\frac{Q}{k} & x = \left[\frac{m}{n}\cdot\frac{1-\alpha}{\alpha}\right]^{\alpha-1}\frac{Q}{k}
\end{array}$$

Since these values of x, y are the only critical point, by assumption the values of

$$x = \frac{Q}{k}\left[\frac{m}{n}\cdot\frac{1-\alpha}{\alpha}\right]^{\alpha-1} y = \frac{Q}{k}\left[\frac{m}{n}\cdot\frac{1-\alpha}{\alpha}\right]^{\alpha}$$

minimize the cost function $C(x,y) = mx+ny$ subject to the constraint of fixed production $Q = kx^\alpha y^{1-\alpha}$.

30.

a)

$$\begin{aligned}
n =& \ 10 \\
\Sigma x_i =& \ 154.20 \\
\Sigma y_i =& \ 80 \\
\Sigma x_i^2 =& \ 2452.18 \\
\Sigma x_i y_i =& \ 1282.74
\end{aligned}$$

Hence,

$$\hat{a} = \frac{(\Sigma x_i^2)(\Sigma y_i) - (\Sigma x_i)(\Sigma x_i y_i)}{n\Sigma x_i^2 - (\Sigma x_i)^2} = \frac{(2452.18)(80) - (154.20)(1282.74)}{10(2452.18) - (154.20)^2} \approx -2.18$$

$$\hat{b} = \frac{n\Sigma x_i y_i - (\Sigma x_i)(\Sigma y_i)}{n\Sigma x_i^2 - (\Sigma x_i)^2} = \frac{10(1282.74) - (154.20)(80)}{10(2452.18) - (154.20)^2} \approx .660$$

So the least squares line is $y = \hat{a} + \hat{b}x \approx -2.18 + .660x$.

b) Substitute $x = 15.0$ into the least squares line: $y = -2.18 + .660(15.0) = 7.72$ feet.

31. We assume the least squares line is $\hat{y} = \hat{b}x$. For the point (x_i, y_i), the vertical deviation from the line $\hat{y} = \hat{b}x$ is:

$$(\hat{b}x_i - y_i).$$

So the sum S of the squares of the vertical deviations is:

$$S = (\hat{b}x_1 - y_1)^2 + (\hat{b}x_2 - y_2)^2 + \cdots + (\hat{b}x_n - y_n)^2.$$

We find $\frac{dS}{d\hat{b}}$:

$$\frac{dS}{d\hat{b}} = 2x_1(\hat{b}x_1 - y_1) + 2x_2(\hat{b}x_2 - y_2) + \cdots + 2x_n(\hat{b}x_n - y_n).$$

Divide by two and set to zero:

$$\hat{b}\sum_{i=1}^{n} x_i^2 - \sum_{i=1}^{n} x_i y_i = 0 \quad \text{(note this is (4) of HP -pg 1008)}$$

We then get $\hat{b} = \frac{\sum_{i=1}^{n} x_i y_i}{\sum_{i=1}^{n} x_i^2}$.

32. a) If we let Dec. '92 represent month 0, then $x_1 = -12$ (representing Dec. '91) and $x_{25} = 12$ (Dec. '93).

Then $\sum_{i=1}^{25} x_i = 0$ and the calculations are simplified.

For $1 \le i \le 25$, let y_i be the closing price for shares of Xerox in the i^{th} month listed.

$$n = 25$$

$$\sum_{i=1}^{25} y_i = 801$$

$$\sum_{i=1}^{25} x_i y_i = 423.75$$

$$\sum_{i=1}^{25} x_i^2 = \sum_{j=-12}^{12} j^2 = 2\sum_{j=1}^{12} j^2 = 2\left(\frac{12(12+1)(2(12)+1)}{6}\right) = 1300$$

(from HP sect. 16.5 pg 815)

$$\begin{aligned}
\hat{b} &= \frac{n\sum_{i=1}^n x_i y_i - (\sum_{i=1}^n x_i)(\sum_{i=1}^n y_i)}{n\sum_{i=1}^n x_i^2 - (\sum_{i=1}^n x_i)^2} \\
&= \frac{25(423.75) - 0(801)}{25(1300) - (0)^2} = \frac{423.75}{1300} \approx .326 \left(= \frac{\sum_{i=1}^n x_i y_i}{\sum_{i=1}^n x_i^2} \right) \\
\hat{a} &= \frac{(\sum_{i=1}^n x_i^2)(\sum_{i=1}^n y_i) - (\sum_{i=1}^n x_i)(\sum_{i=1}^n x_i y_i)}{n\sum_{i=1}^n x_i^2 - (\sum_{i=1}^n x_i)^2} \\
&= \frac{(1300)(801) - 0(423.75)}{25(1300) - (0)^2} \\
&= \frac{801}{25} = 32.04 \left(= \frac{\sum_{i=1}^n y_i}{n} \right)
\end{aligned}$$

So the least squares line is: $y = .326x + 32.04$ (where the 0^{th} month is Dec. '92).

b) Feb. '96 is $3(12) + 2 = 38$ months after Dec. '92 and hence $x = 38$. So the expected closing price for Xerox stocks in Feb. '96 is:

$$.326(38) + 32.04 = 44.43 \approx 44\frac{3}{8} \quad \text{or} \quad 44\frac{1}{2}$$

(rounding to the nearest eighth, which is how stock prices are quoted).