Supplementary Questions for HP Chapter 17

1. Sketch graphs of the following functions:

(a) $z = \frac{y}{x}$

(b) $z = ye^{-x}$

(c) $z = x^2 - y^2$

2. Find the domain and range of the given functions:

(a) $f(x, y) = \sqrt{36 - 9x^2 - 4y^2}$

(b) $f(x, y, z) = x^2 \ln(x - y + z)$

(c) $f(x, y, z) = \frac{x}{yz}$

(c) $f(x, y, z) = \frac{1}{\sqrt{x^2+y^2+z^2-1}}$

3. The Ideal Gas Law $PV = nRT$ (n is the number of moles of the gas, R is a constant) determines each of the three variables P, V, and T (pressure, volume, and absolute temperature, respectively) as functions of the other two. Show that

$$\frac{\partial P}{\partial V} \cdot \frac{\partial V}{\partial T} \cdot \frac{\partial T}{\partial P} = -1.$$

4. There is only one point at which the tangent plane to the surface $f(x, y) = x^2 + 2xy + 10y - 6x$ is horizontal. Find it.

Note: A plane $g(x, y) = Ax + By + C$ is tangent to a surface $f(x, y)$ at the point $(x_0, y_0, f(x_0, y_0))$ if $f_x(x_0, y_0) = g_x(x_0, y_0)$, $f_y(x_0, y_0) = g_y(x_0, y_0)$, and $f(x_0, y_0) = g(x_0, y_0)$.

5. Find $f_x(x, y)$ and $f_y(x, y)$ of the following:

(a) $f(x, y) = \int_x^y e^{t^2} dt$

(b) $f(x, y) = \int_y^x \frac{e^t}{t} dt$

6. The differential of a function of several variables $f(x_1, x_2, \ldots, x_n)$ is

$$df = \frac{\partial f}{\partial x_1}(x_1, \ldots, x_n)dx_1 + \frac{\partial f}{\partial x_2}(x_1, \ldots, x_n)dx_2 + \ldots + \frac{\partial f}{\partial x_n}(x_1, \ldots, x_n)dx_n$$

The differential formula for approximation in several variables is given by

$$f(x_1 + dx_1, x_2 + dx_2, \ldots, x_n + dx_n) \approx f(x_1, x_2, \ldots, x_n) + df.$$

Use differentials to approximate $(1.98)^3\sqrt{(3.01)^2 + (3.97)^2}$.

7. Just as we can represent price in terms of quantity, we can also represent quantity in terms of price. $Q_1 = Q_1(P_1, P_2, I)$ represents the demand for good 1 in terms of the prices of goods 1 and 2 and income. There are three elasticities that are useful:

1) *Own Price Elasticity of Demand:*

This is
$$\varepsilon_1 = \frac{\%\text{ change in demand for good 1}}{\%\text{ change in price of good 1}} = \frac{\Delta Q_1/Q_1}{\Delta P_1/P_1} = \frac{P_1}{Q_1}\cdot\frac{\Delta Q_1}{\Delta P_1}$$

In terms of calculus, $\varepsilon_1 = \frac{P_1 \cdot \frac{\partial Q_1}{\partial P_1}(P_1,P_2,I)}{Q_1(P_1,P_2,I)}$

2) *Cross Price Elasticity of Demand:*

This is $\varepsilon_{Q_1,P_2} = \frac{\%\text{ change in demand for good 1}}{\%\text{ change in price of good 2}}$

so $\varepsilon_{Q_1,P_2} = \frac{P_2 \cdot \frac{\partial Q_1}{\partial P_2}(P_1,P_2,I)}{Q_1(P_1,P_2,I)}$

3) *Income Elasticity of Demand:*

This is $\varepsilon_{Q_1,I} = \frac{\%\text{ change in demand for good 1}}{\%\text{ change in income}}$

so $\varepsilon_{Q_1,I} = \frac{I \cdot \frac{\partial Q_1}{\partial I}(P_1,P_2,I)}{Q_1(P_1,P_2,I)}$

The demand function $Q_1 = K_1 P_1^{a_{11}} P_2^{a_{12}} I^{b_1}$ is called a *constant elasticity demand function.*

Compute the three elasticities (own price, cross price, and income) and show that they are all constants.

8. A firm has the Cobb-Douglas production function (see Ex. 6, pg 970 of HP) $f(x_1, x_2, x_3) = 10x_1^{\frac{1}{3}}, x_2^{\frac{1}{2}}, x_3^{\frac{1}{6}}$. Currently, it is using the input bundle $(27, 16, 64)$ (i.e., $x_1 = 27, x_2 = 16, x_3 = 64$).

(a) How much is it producing?

(b) Find $f_{x_i}(x_1, x_2, x_3)$ for $i = 1, 2, 3$. Use these to estimate the firm's new output when x_1 increases to 27.1, x_2 decreases to 15.7, and x_3 remains the same.

(c) Use a calculator to compare your answer in part (b) with the actual output.

(d) Do (b) and (c) for $\Delta x_1 = \Delta x_2 = 0.2$ and $\Delta x_3 = -0.4$.

9. The price of a bond is given by

$$P = v(1+i)^{-n} + rVa_{\overline{n}|i} \;=\; V(1+i)^{-n} + rV\left(\frac{1-(1+i)^{-n}}{i}\right).$$

(a) Find $\frac{\partial P}{\partial n}$, for fixed V, i, r, and show that for fixed yield i and fixed coupon rate r, the price of the bond

i) increases as n decreases, if $i > r$

ii) decreases as n decreases, if $i < r$

iii) is constant with respect to n if $i = r$

(b) Find $\frac{\partial P}{\partial i}$, for fixed $V, r, n (n \geq 2)$, and show that price decreases as the yield increases. You may assume that $(1+i)^n > 1 + ni$.

10. Find $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$ if z is defined implicitly as a function of x and y by $xy^2z^3 + x^3y^2z \;=\; x + y + z$.

11. If z is defined implicitly by the relation

$$z\ln(x^2+y^2) \;=\; 1,$$

find $x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y}$ in terms of z.

12. According to Van der Waals' equation, 1 mole of a gas satisfies the equation:

$$(P + \frac{a}{V^2})(V-b) \;=\; 82.06T$$

where P is the pressure (in atmospheres), V is the volume (in cubic centimetres) and T is the absolute temperature (in Kelvin (K), where $K \approx^0 C + 273$). For carbon dioxide, $a = 3.59 \times 10^6$ and $b = 42.7$, and V is 25,600 cm^3 when P is 1 atm. and $T = 313K$.

(a) Compute $\frac{\partial V}{\partial P}$ by differentiating van der Waals' equation with T held constant. Then estimate the change in volume that would result from an increase of 0.1 atm of pressure with T at 313K.

(b) Compute $\frac{\partial V}{\partial T}$ by differentiating van der Waals' equation with P held constant. Then estimate the change in volume that would result from an increase of $1K$ in temperature with P held at 1 atm.

13. If $z^3 - xz - y = 0$, prove that $\frac{\partial^2 z}{\partial x \partial y} \;=\; -\frac{(3z^2+x)}{(3z^2-x)^3}$

14. *A case where mixed partials are not equal.* Let

$$f(x,y) \;=\; \begin{cases} \frac{x^3y - xy^3}{x^2+y^2} & \text{if } (x,y) \neq (0,0) \\ 0 & \text{if } (x,y) = (0,0) \end{cases}$$

(a) Find $f_x(x,y)$ and $f_y(x,y)$ when $(x,y) \neq (0,0)$

(b) For a function $f(x,y)$, the actual definitions of $f_x(x,y)$ and $f_y(x,y)$ are, using limits:

$$f_x(a,b) := \lim_{h\to 0} \frac{f(a+h,b) - f(a,b)}{h}$$

$$f_y(a,b) := \lim_{h\to 0} \frac{f(a,b+h) - f(a,b)}{h}$$

Using these definitions, find $f_x(0,0)$ and $f_y(0,0)$

(c) Show that $f_{xy}(0,0) = -1$ and $f_{yx}(0,0) = 1$.

15. A steady-state temperature function $u(x,y)$ for a thin flat plate satisfies LaPlace's equation:

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0.$$

Determine which of the following functions satisfies LaPlace's equation:

(a) $u = \ln(\sqrt{x^2+y^2})$

(b) $u = \sqrt{x^2+y^2}$

16. If $u = e^{\sum_{i=1}^n a_i x_i}$, where $\sum_{i=1}^n a_i^2 = 1$, show that:

$$\sum_{i=1}^n \frac{\partial^2 u}{\partial x_i^2} = u.$$

17. Show that the function $z = xe^y + ye^x$ is a solution of the equation:

$$\frac{\partial^3 z}{\partial x^3} + \frac{\partial^3 z}{\partial y^3} = x\frac{\partial^3 z}{\partial x \partial y^2} + y\frac{\partial^3 z}{\partial x^2 \partial y}.$$

18.

(a) Consider the function $U(r,s)$. If $x = 2r - s$ and $y = r + 2s$, find $\frac{\partial^2 U}{\partial y \partial x}$ in terms of derivatives with respect to r and s.

Hint: Find r and s in terms of x and y. Then, for example, $\frac{\partial U}{\partial x} = \frac{\partial U}{\partial r}\frac{\partial r}{\partial x} + \frac{\partial U}{\partial s}\frac{\partial s}{\partial x}$.

(b) Let $U(r,s) = r^2 e^{rs}$. As above, $x = 2r - s$ and $y = r + 2s$. Find $\frac{\partial^2 U}{\partial y \partial x}$.

19. If $z = y + f(x^2 - y^2)$ where f is a differentiable function of one variable, show that

$$y\frac{\partial z}{\partial x} + x\frac{\partial z}{\partial y} = x.$$

20. Show that $Y = f(x+at) + g(x-at)$ satisfies $\frac{\partial^2 Y}{\partial t^2} = a^2\left(\frac{\partial^2 Y}{\partial x^2}\right)$, where f and g are assumed to be at least twice differentiable and a is any constant.

21. Suppose that $w(x,y) = f(u,v,x,y)$, where u and v are functions of x and y. Here x and y play dual roles as both intermediate and independent variables.

The chain rule yields

$$\begin{aligned}\frac{\partial w}{\partial x} &= \frac{\partial f}{\partial u}\cdot\frac{\partial u}{\partial x} + \frac{\partial f}{\partial v}\cdot\frac{\partial v}{\partial x} + \frac{\partial f}{\partial x}\cdot\frac{\partial x}{\partial x} + \frac{\partial f}{\partial y}\cdot\frac{\partial y}{\partial x} \\ &= \frac{\partial f}{\partial u}\cdot\frac{\partial u}{\partial x} + \frac{\partial f}{\partial v}\cdot\frac{\partial v}{\partial x} + \frac{\partial f}{\partial x}\end{aligned}$$

(since $\frac{\partial x}{\partial x} = 1$ and $\frac{\partial y}{\partial x} = 0 - y$ is independent of x).

Similarly, $\frac{\partial w}{\partial y} = \frac{\partial f}{\partial u}\cdot\frac{\partial u}{\partial y} + \frac{\partial f}{\partial v}\cdot\frac{\partial v}{\partial y} + \frac{\partial f}{\partial y}$. For the following questions, find $\frac{\partial w}{\partial x}$ and $\frac{\partial w}{\partial y}$ as functions of x and y.

(a) $f = \sqrt{uvxy}$; $u = \sqrt{x-y}$, $v = \sqrt{x+y}$

(b) $f = uv - xy$; $u = \frac{x}{x^2+y^2}$, $v = \frac{y}{x^2+y^2}$

22. Find and classify all critical points of

$$f(x,y) = x^4 - 8xy + 2y^2 - 3.$$

23. Find the maximum volume of a rectangular box that can be sent from a post office if the sum of its length and girth cannot exceed 108 inches.

Note: Here, girth represents the perimeter of the box using the height and width co-ordinates, i.e. it is the perimeter of a "slice" of the rectangular box when we temporarily fix the length co-ordinate at some value ℓ_0.

24. Let us postulate a two-product firm under circumstances of pure competition. This implies that prices must be kept fixed and hence P_1 and P_2 (the prices of products one and two respectively) are not in the firm's control and are considered constant. The firm's cost function is assumed to be:

$$C = 2Q_1^2 + Q_1Q_2 + 2Q_2^2$$

where $Q_i (i = 1, 2)$ is the quantity of item i produced (and, for the sake of simplicitly, sold as well).

Find the values for Q_1 and Q_2 in terms of P_1 and P_2 where the firm's profit is maximized.

25. Let us now transplant the previous problem into the setting of a monopolistic market. By virtue of this new market structure assumption, the prices of the two products will now vary with their output levels (again, this is assumed to be identical with the sales level). The prices P_1 and P_2 are related to Q_1 and Q_2 as follows:

$$\begin{aligned} P_1 &= 55 - Q_1 - Q_2 \\ P_2 &= 70 - Q_1 - 2Q_2. \end{aligned}$$

We again assume the total-cost function to be:

$$C = Q_1^2 + Q_1Q_2 + Q_2^2.$$

Find the values of Q_1 and Q_2 where the firm's profit is maximized. Determine the optimal prices from this.

26. Find the maximum and minimum values of xyz subject to the constraint $x^2 + y^2 + z^2 = 3$.

27. Find the shortest distance from the origin to the hyperbola $x^2+8xy+7y^2 = 225,\ z = 0$. Hint: The distance from the origin to a point (x, y) is $\sqrt{x^2 + y^2}$.

28. Find the critical points of the following functions subject to the given constraints:

1) $f(x, y) = \frac{1}{x} + \frac{1}{y};\ \frac{1}{x^2} + \frac{1}{y^2} = 1$
2) $f(x, y, z) = x + y + z;\ \frac{1}{x} + \frac{1}{y} + \frac{1}{z} = 1.$

29. Suppose a production function is given by the Cobb-Douglas function $q = kx^\alpha y^{1-\alpha}$ where q is the production, $k > 0$ is a constant, $0 < \alpha < 1$ is a constant, x is labour and y is capital.

Suppose further the production is fixed at a constant $Q > 0$, i.e. $Q = kx^\alpha y^{1-\alpha}$. What values of x and y minimize the cost function $C(x, y) = mx + ny$ where $m, n > 0$ are constants? You may assume that the only critical point is a minimum.

30. The paper "Effects of Bike Lanes on Driver and Bicyclist Behavior" (ASCE Transportation Eng. J., 1977, pp. 243-256) reported the results of a regression analysis with x = available travel space in feet (a convenient measure of roadway width, defined as the distance between a cyclist and the roadway center line) and separation distance y between a bike and a passing car (determined by photography). The data, for 10 streets with bike lanes, appears below

x :	12.8	12.9	12.9	13.6	14.5	14.6	15.1	17.5	19.5	20.8
y :	5.5	6.2	6.3	7.0	7.8	8.3	7.1	10.0	10.8	11.0

(a) Find the least squares line.

(b) What separation distance would you predict for another street that has 15.0 as its available travel space value?

Note: in the above data there are two entries for $x = 12.9$. This will not cause any problems (although it does guarantee that the scatter diagram is not a line!). Just treat the two separate entries of $x = 12.9$ as two separate data points (i.e. when calculating Σx_i, add 12.9 twice).

31. Suppose an investigator has data on the amount of shelf space x devoted to display of a particular product and sales revenue y for that product. Then the investigator may wish to fit a model for which the true regression line passes through (0,0). The appropriate model is $\hat{y} = \hat{b}x$. Assume that $(x_1, y_1), (x_2, y_2), \ldots, (x_n, y_n)$ are observed pairs generated from this model. Derive the linear regression coefficient for $\hat{b}$ using the method of least squares.

Hint: Imitate the strategy in HP, pg 1008 by setting $\frac{dS}{d\hat{b}} = 0$. Note that we are setting $\hat{a} = 0$.

32.

(a) The following is a list of monthly TSE closings for shares of Xerox Canada from December of 1991 until December of 1993 inclusive. Find the least squares line, using any simplifications that you may wish.

(b) What is the expected monthly closing price of Xerox stocks in Feb. '96?

December 1991 and 1992			**1993**
Month	**Stock Price**	**Month**	**Stock Price**
Dec. '91	$25\frac{6}{8}$	Jan. '93	$35\frac{3}{8}$
Jan. '92	$30\frac{1}{8}$	Feb. '93	$33\frac{6}{8}$
Feb. '92	$30\frac{6}{8}$	Mar. '93	$35\frac{4}{8}$
Mar. '92	29	Apr. '93	33
Apr. '92	$29\frac{2}{8}$	May '93	$31\frac{4}{8}$
May '92	$29\frac{6}{8}$	Jun. '93	$34\frac{4}{8}$
Jun. '92	$26\frac{6}{8}$	Jul. '93	31
Jul. '92	$29\frac{4}{8}$	Aug. '93	32
Aug. '92	30	Sep. '93	$32\frac{4}{8}$
Sep. '92	$32\frac{6}{8}$	Oct. '93	$33\frac{6}{8}$
Oct. '92	$30\frac{5}{8}$	Nov. '93	37
Nov. '92	$33\frac{4}{8}$	Dec. '93	40
Dec. '92	$33\frac{3}{8}$		