Solutions to Supplementary Questions for HP Chapter 15

1. (a) Let $u = x + 10$

$du = dx$.

$$\text{Then } \int \ln(x+10)dx = \int \ln(u)du$$
$$= u[\ln(u) - 1] + C_1$$
$$= u\ln(u) - u + C_1$$
$$= (x+10)\ln(x+10) - x - 10 + C_1$$
$$= (x+10)\ln(x+10) - x + C_2$$

C_1, C_2 are arbitrary constants.

(b) Let $u = \ln(x+10)$ $dv = dx$

$du = \frac{1}{x+10}dx$ $v = x$

Then $\int \ln(x+10)dx = x\ln(x+10) - \int \frac{x}{x+10}dx$

Now substitute $w = x + 10$

$dw = dx$

$x = w - 10$

$$\int \ln(x+10)dx = x\ln(x+10) - \int \frac{w-10}{w}dw$$
$$= x\ln(x+10) - \int dw + \int \frac{10}{w}dw$$
$$= x\ln(x+10) - w + 10\ln w + C_1$$
$$= x\ln(x+10) - x - 10 - 10\ln(x+10) + C_1$$
$$= (x+10)\ln(x+10) - x + C_2$$

(c)

Let $u = \ln(x+10)$ $dv = dx$

$du = \frac{1}{x+10}dx$ $v = x + 10$

$$\text{Then } \int \ln(x+10)dx = (x+10)\ln(x+10) - \int dx$$
$$= (x+10)\ln(x+10) - x + C$$

2. First, we must ensure that xe^{3x} and $\frac{2}{3}xe^{x^2}$ do not intersect in between $x = 0$ and $x = 3$:

$$xe^{3x} = \frac{2}{3}xe^{x^2} \Rightarrow x(\frac{2}{3}e^{x^2} - e^{3x}) = 0$$
$$\Rightarrow x = 0 \text{ or } \frac{2}{3}e^{x^2} = e^{3x}$$

The latter case gives $\ln\frac{2}{3} + x^2 = 3x$

$$\Rightarrow x^2 - 3x + \ln(\frac{2}{3}) = 0$$

Noting that $\ln\frac{2}{3} \approx -.4055$, we get

$$x \approx \frac{-(-3) \pm \sqrt{(-3)^2 - 4(-.4055)}}{2} \Rightarrow x \approx 3.130 \text{ or } x \approx -0.129$$

And hence, since for $x = 1$, $(1)e^{3(1)} > \frac{2}{3}(1)e^{(1)^2}$ and since $xe^{3x} \neq \frac{2}{3}xe^{x^2}(0 < x < 3)$ then $xe^{3x} \geq \frac{2}{3}xe^{x^2}(0 \leq x \leq 3)$. Now we just calculate $\int_0^3 (xe^{3x} - \frac{2}{3}xe^{x^2})dx = \int_0^3 xe^{3x}dx - \frac{2}{3}\int_0^3 xe^{x^2}dx$.

(i) Solving $\int_0^3 xe^{3x}dx$: This is by parts. Let $u = x \qquad v = \frac{1}{3}e^{3x}$

$du = dx \quad dv = e^{3x}dx$

Now, $\int_1^b udv = uv\Big|_a^b - \int_a^b vdu$, and so $\int_0^3 xe^{3x}dx = \frac{1}{3}xe^{3x}\Big|_0^3 - \int_0^3 \frac{1}{3}e^{3x}dx$. So

$$\int_0^3 xe^{3x}dx = (\frac{3}{3}e^9 - 0) - \frac{1}{9}e^{3x}\Big|_0^3$$
$$= e^9 - (\frac{e^9 - 1}{9}) = \frac{8e^9 + 1}{9}$$

(ii) Also $$\int_0^3 xe^{x^2}dx = \frac{e^{x^2}}{2}\Big|_0^3 = \frac{e^9 - 1}{2}$$

So $$\int_0^3 xe^{3x}dx - \frac{2}{3}\int_0^3 xe^{x^2}dx$$
$$= \frac{8e^9 + 1}{9} - \frac{2}{3}(\frac{e^9 - 1}{2}) = \frac{5e^9 + 4}{9}$$

3. (a) Let $u = x^n \qquad dv = e^x dx$

$du = nx^{n-1}dx, \quad v = e^x$

Then $\int udv = uv - \int vdu$

so

$$\begin{aligned}\int x^n e^x dx &= x^n e^x - \int e^x n x^{n-1} dx \\ &= x^n e^x - n\int x^{n-1} e^x dx\end{aligned}$$

(b)

$$\begin{aligned}\int x^5 e^x dx &= x^5 e^x - 5\int x^4 e^x dx \\ &= x^5 e^x - 5(x^4 e^x - 4\int x^3 e^x dx) \\ &= x^5 e^x - 5x^4 e^x + 20(x^3 e^x - 3\int x^2 e^x dx) \\ &= x^5 e^x - 5x^4 e^x + 20x^3 e^x - 60(x^2 e^x - 2\int e^x dx) \\ &= x^5 e^x - 5x^4 e^x + 20x^3 e^x - 60x^2 e^x + 120(xe^x - \int e^x dx) \\ &= x^5 e^x - 5x^4 e^x + 20x^3 e^x - 60x^2 e^x + 120xe - 120e^x + C\end{aligned}$$

4. We are forgetting about the arbitrary constant, so we cannot subtract $\int \frac{1}{x} dx$ from both sides of the equation $\int \frac{1}{x} dx = 1 + \int \frac{1}{x} dx$. In fact, we have $\ln|x| + C_1 = 1 + \ln|x| + C_2$ where C_1, C_2 are arbitrary. Clearly, choosing one arbitrary constant determines the other. For example, if $C_1 = 0$ then $C_2 = -1$. If there are limits involved, then we get

$$\begin{aligned}\int_b^a \frac{1}{x} dx &= 1\Big|_b^a + \int_b^a \frac{1}{x} dx \\ &= 1 - 1 + \int_b^a \frac{1}{x} dx \\ &= \int_b^a \frac{1}{x} dx\ .\end{aligned}$$

5.

$$\begin{aligned}P.V. &= \int_0^n (1000 + 6t)e^{-rt} dt \\ &= \int_0^n (1000e^{-rt} dt + \int_0^n 6te^{-rt} dt\end{aligned}$$

For the second integral we integrate by parts; $u = 6t$, $dv = e^{-rt}$.

$$
\begin{aligned}
P.V. &= -\frac{1000}{r}e^{-rt}\Big|_0^n + -\frac{6t}{r}e^{-rt}\Big|_0^n + \int_0^n \frac{6}{r}e^{-rt}dt \\
P.V. &= -\frac{1000}{r}e^{-rt} - \frac{6t}{r}e^{-rt} - \frac{6}{r^2}e^{-rt}\Big|_0^n \\
P.V. &= e^{-rt}(-\frac{1000}{r} - \frac{6t}{r} - \frac{6}{r^2})\Big|_0^n \\
P.V. &= -e^{-rn}(\frac{1000}{r} + \frac{6n}{r} + \frac{6}{r^2}) + (\frac{1000}{r} + \frac{6}{r^2})
\end{aligned}
$$

We have the formula

$$P.V. = (\frac{1000}{r} + \frac{6}{r^2}) - (\frac{1000}{r} + \frac{6n}{r} + \frac{6}{r^2})e^{-rn}$$

(a) $n = 2$, $r = 0.05$ gives

$$
\begin{aligned}
P.V. &= (\frac{1000}{0.05} + \frac{6}{(0.05)^2}) - (\frac{1000}{0.05} + \frac{12}{0.05} + \frac{6}{(0.05)^2})e^{-0.1} \\
&= \$1914.48
\end{aligned}
$$

(b) $n = 2$, $r = 0.10$ gives

$$
\begin{aligned}
P.V. &= (\frac{1000}{0.1} + \frac{6}{(0.1)^2}) - (\frac{1000}{0.1} + \frac{12}{0.1} + \frac{6}{(0.1)^2})e^{-0.2} \\
&= \$1823.21
\end{aligned}
$$

(c) $n = 3$, $r = 0.05$ gives

$$
\begin{aligned}
P.V. &= (\frac{1000}{0.05} + \frac{6}{(0.05)^2}) - (\frac{1000}{0.05} + \frac{18}{0.05} + \frac{6}{(0.05)^2})e^{-0.15} \\
&= \$2810.29
\end{aligned}
$$

6. (a) Let $u = \sqrt[6]{x} = x^{1/6}$

$$du = \frac{1}{6x^{5/6}}dx = \frac{1}{6(x^{1/6})^5}dx = \frac{1}{6u^5}dx \text{ so } 6u^5 du = dx.$$

Also, $u = x^{1/6}$ so $u^2 = x^{1/3} = \sqrt[3]{x}$ and $u^3 = x^{1/2} = \sqrt{x}$.

Therefore, $\displaystyle\int \frac{dx}{\sqrt{x} - \sqrt[3]{x}} = 6\int \frac{u^5}{u^3 - u^2}du = 6\int \frac{u^3}{u-1}du.$

Now we do the long division

$$\begin{array}{r} u^2 + u + 1 \\ u - 1 \overline{\smash{)}\, u^3 } \\ \underline{u^3 - u^2} \\ u^2 \\ \underline{u^2 - u} \\ u \\ \underline{u - 1} \\ 1 \end{array}$$

so

$$\begin{aligned} 6\int \frac{u^3}{u-1}du &= 6\int (u^2 + u + 1 + \frac{1}{u-1})du \\ &= 6\left[\frac{u^3}{3} + \frac{u^2}{2} + u + \ln|u-1|\right] + C \\ &= 2u^3 + 3u^2 + 6u + 6\ln|u-1| + c \\ &= 2\sqrt{x} + 3x^{1/3} + 6x^{1/6} + 6\ln|x^{1/6} - 1| + C \end{aligned}$$

(b) $\displaystyle\int \frac{\sqrt{x+4}}{x}dx$

Let $u = \sqrt{x+4}$

$du = \dfrac{1}{2\sqrt{x+4}}dx = \dfrac{1}{2u}dx$ so $2udu = dx$.

Also, $u = \sqrt{x+4}$ so $u^2 = x + 4$ and $x = u^2 - 4$.

Therefore, $\displaystyle\int \frac{\sqrt{x+4}}{x}dx = 2\int \frac{u^2}{u^2-4}du$

We now do long division:

$$\begin{array}{r} 1 \\ u^2 - 4 \overline{\smash{)}\, u^2 } \\ \underline{u^2 - 4} \\ 4 \end{array}$$

so

$$\begin{aligned} \int \frac{\sqrt{x+4}}{x}dx = 2\int \frac{u^2}{u^2-4}du &= 2\int 1 + \frac{4}{u^2-4}du \\ &= 2\int \left[1 + \frac{4}{(u-2)(u+2)}\right] du \end{aligned}$$

Write

$$\begin{aligned} \frac{4}{(u-2)(u+2)} &= \frac{A}{u-2} + \frac{B}{u+2} \\ 4 &= A(u+2) + B(u-2) \\ A + B = 0 &\text{ and } 2A - 2B = 4 \\ A = -B &\text{ and } -4B = 4 \text{ so } B = -1, A = 1 \end{aligned}$$

Therefore

$$\begin{aligned}\int \frac{\sqrt{x+4}}{x}dx &= 2\int \left[1+\frac{1}{u-2}-\frac{1}{u+2}\right]du \\ &= 2u + 2\ln|u-2| - 2\ln|u+2| + C \\ &= 2u + 2\ln\left|\frac{u-2}{u+2}\right| + C \\ &= 2\sqrt{x+4} + 2\ln\left|\frac{\sqrt{x+4}-2}{\sqrt{x+4}+2}\right| + C\end{aligned}$$

7. We write

$$\begin{aligned}\frac{(x-1)^{n+1}-(x+1)^{n+1}}{(x^2-1)^{n+1}} &= \frac{(x-1)^{n+1}-(x+1)^{n+1}}{[(x+1)(x-1)]^{n+1}} \\ &= \frac{(x-1)^{n+1}}{(x+1)^{n+1}(x-1)^{n+1}} - \frac{(x+1)^{n+1}}{(x+1)^{n+1}(x-1)^{n+1}} \\ &= \frac{1}{(x+1)^{n+1}} - \frac{1}{(x-1)^{n+1}}\end{aligned}$$

so

$$\begin{aligned}\int \frac{(x-1)^{n+1}-(x+1)^{n+1}}{(x^2-1)^{n+1}}dx &= \int \frac{1}{(x+1)^{n+1}} - \int \frac{1}{(x-1)^{n+1}}dx \\ &= -\frac{1}{n(x+1)^n} + \frac{1}{n(x-1)^n} + C \\ &= \frac{-(x-1)^n+(x+1)^n}{n(x+1)^n(x-1)^n} + C \\ &= -\frac{1}{n}\left[\frac{(x-1)^n-(x+1)^n}{(x^2-1)^n}\right] + C\end{aligned}$$

8. (a) $\int \frac{e^{4t}}{(e^{2t}-1)^3}dt = \int \frac{e^{2t}e^{2t}}{(e^{2t}-1)^3}dt$

Let $\quad u = e^{2t}-1$

$$du = 2e^{2t}dt$$

$$\frac{1}{2}du = e^{2t}dt$$

$$e^{2t} = u+1$$

so $\int \frac{e^{4t}}{(e^{2t}-1)^3}dt = \frac{1}{2}\int \frac{u+1}{u^3}du$

Let $\frac{u+1}{u^3} = \frac{A}{u^3} + \frac{B}{u^2} + \frac{C}{u}$

$$u+1 = A + Bu + Cu^2$$

so $A = 1,\ B = 1,\ C = 0$

Therefore,

$$\begin{aligned}\int \frac{e^{4t}}{(e^{2t}-1)^3}dt &= \frac{1}{2}\int (\frac{1}{u^3} + \frac{1}{u^2})du \\ &= \frac{1}{2}[-\frac{1}{2u^2} - \frac{1}{u}] + C \\ &= \frac{1}{2}[\frac{-1-2u}{2u^2}] + C \\ &= \frac{-1-2(e^{2t}-1)}{4(e^{2t}-1)^2} + C \\ &= \frac{1-2e^{2t}}{4(e^{2t}-1)^2} + C\end{aligned}$$

(b) Let $u = 3 + 2\ln t \quad u - 1 = 2 + 2\ln t$

$du = \frac{2}{t}dt \qquad \frac{u-1}{2} = 1 + \ln t$

$\frac{1}{2}du = \frac{1}{t}dt$

Therefore, $\int \frac{1+\ln t}{t(3+2\ln t)^2}dt = \frac{1}{4}\int \frac{u-1}{u^2}du$

Let $\frac{u-1}{u^2} = \frac{A}{u} + \frac{B}{u^2}$

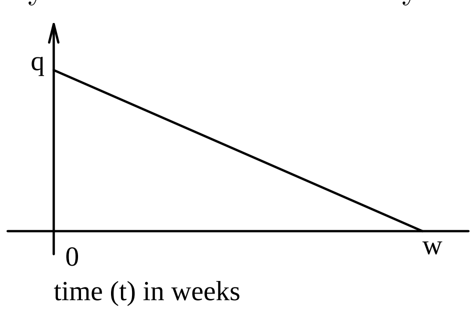
$u - 1 = Au + B$

so $A = 1,\ B = -1$

Therefore,

$$\begin{aligned}\int \frac{1+\ln t}{t(3+2\ln t)^2}dt &= \frac{1}{4}\int (\frac{1}{u} - \frac{1}{u^2})du \\ &= \frac{1}{4}[\ln|u| + \frac{1}{u}] + C \\ &= \frac{1}{4}[\ln|3+2\ln t| + \frac{1}{(3+2\ln t)}] + C\ .\end{aligned}$$

9. Since the inventory level I declines at a steady rate, it may be described by the graph



Thus the inventory level is given by

$$(I - q) = -\frac{q}{w}(t - 0)$$
$$I = (-\frac{q}{w})t + q$$

The average inventory level over $[0, w]$ is

$$\frac{1}{w-0}\int_0^w -\frac{q}{w}t + q\,dt = \frac{1}{w}[-\frac{qt^2}{2w} + qt\Big|_0^w]$$
$$= \frac{1}{w}[-\frac{qw}{2} + qw]$$
$$= q - \frac{q}{2}$$
$$= \frac{q}{2}$$

Since the product is sold at a steady rate, it is expected by common sense that the average inventory level is half the original inventory level.

10. (a) The average value is

$$\frac{1}{2}\int_0^2 x(x+1)dx = \frac{1}{2}\int_0^2 x^2 + xdx$$
$$= \frac{1}{2}[\frac{1}{3}x^3 + \frac{1}{2}x^2\Big|_0^2]$$
$$= \frac{4}{3} + 1 = \frac{7}{3}$$

We wish so solve $c(c+1) = \frac{7}{3}$

$$c^2 + c - \frac{7}{3} = 0$$

Using the quadratic formula,

$$c = \frac{-1 \pm \sqrt{\frac{31}{3}}}{2} \approx -2.11, 1.11$$

The answer is $c = \dfrac{-1 + \sqrt{\frac{31}{3}}}{2}$.
Since it is the only number of the above two numbers that lies between 0 and 2.

(b) The average value is

$$\frac{1}{e-1}\int_1^e \frac{1}{x} - \frac{1}{x^2}dx = \frac{1}{e-1}[\ln|x| + \frac{1}{x}|_1^e]$$
$$= \frac{1}{e-1}[1 + \frac{1}{e} - 0 - 1]$$
$$= \frac{1}{e(e-1)}$$

We wish to solve

$$\frac{1}{c} - \frac{1}{c^2} = \frac{1}{e(e-1)}$$
$$c - 1 = \frac{c^2}{e(e-1)}$$
$$\frac{1}{e(e-1)}c^2 - c + 1 = 0$$

Using the quadratic formula,

$$c = \frac{1 \pm \sqrt{1 - \frac{4}{e(e-1)}}}{(\frac{2}{e(e-1)})}$$
$$= \frac{e(e-1)}{2}[1 \pm \sqrt{1 - \frac{4}{e(e-1)}}]$$
$$\approx 1.45, 3.22$$

The answer is $c = \frac{e(e-1)}{2}\left[1 - \sqrt{1 - \frac{4}{e(e-1)}}\ \right]$
since this is the only one of the above two numbers lying between 1 and e.

11. Separating variables, we have

$$\frac{dx}{(30-x)(20-x)} = k\,dt \qquad\qquad \int \frac{1}{(30-x)(20-x)}dx = \int k\,dt$$

This integration on the left may be done by partial fractions.

$$\frac{1}{(30-x)(20-x)} = \frac{A}{30-x} + \frac{B}{20-x}$$
$$1 = A(20-x) + B(30-x)$$
$$20A + 30B = 1$$
$$-A - B = 0 \Rightarrow A = -B$$
$$\Rightarrow 30B - 20B = 1$$
$$10B = 1$$
$$B = \frac{1}{10}\ ,\ A = -\frac{1}{10}$$

Therefore,

$$\frac{1}{10}\int \frac{1}{20-x} - \frac{1}{30-x}dx = \int kdt$$

$$\frac{1}{10}(-\ln|20-x| + \ln|30-x|) = kt + C\ ,\ C \text{ is an arbitrary constant}$$

$$= \frac{1}{10}\ln|\frac{30-x}{20-x}| = kt + C$$

$$= \ln(\frac{30-x}{20-x}) = 10kt + 10C \text{ (since} x < 20\,,\ \frac{30-x}{20-x} > 0)$$

Since $x(0) = 0$,

$$\ln(\frac{30}{20}) = 10C$$

$$\ln(\frac{3}{2}) = 10C$$

so we now have $\ln(\frac{30-x}{20-x}) = 10kt + \ln(\frac{3}{2})$.

Taking both sides to the power of e, we have

$$\frac{30-x}{20-x} = e^{10kt+\ln(3/2)}$$

$$\frac{30-x}{20-x} = e^{\ln(3/2)}e^{10kt}$$

$$\frac{30-x}{20-x} = \frac{3}{2}e^{10kt}$$

$$30 - x = 30e^{10kt} - \frac{3x}{2}e^{10kt}$$

$$x - \frac{3x}{2}e^{10kt} = 30 - 30e^{10kt}$$

$$x(1 - \frac{3}{2}e^{10kt}) = 30(1 - e^{10kt})$$

$$x = \frac{30(1-e^{10kt})}{(\frac{2-3e^{10kt}}{2})}$$

$$x = \frac{60(1-e^{10kt})}{2-3e^{10kt}}$$

12. (a)

$$\frac{dy}{dx} = e^{x-y} = e^x e^{-y}$$

$$\text{so } e^y dy = e^x dx$$

$$\int e^y dy = \int e^x dx$$

$$e^y = e^x + C\ ,\ C \text{ is an arbitrary constant}$$

$$y = \ln(e^x + C)$$

Since $y(0) = 1$,

$$
\begin{aligned}
1 &= \ln(1 + C) \\
e &= 1 + C \\
C &= e - 1
\end{aligned}
$$

Therefore, $y = \ln(e^x + e - 1)$ is a solution.

(b) $\dfrac{dy}{dx} = \dfrac{0.2y(18 + 0.1x)}{x(100 + 0.5y)}$

$(\dfrac{100 + 0.5y}{y})dy = \dfrac{0.2(18 + 0.1x)}{x}dx$

$(\dfrac{100}{y} + 0.5)dy = (\dfrac{3.6}{x} + 0.02)dx$

$\displaystyle\int(\frac{100}{y} + 0.5)dy = \int(\frac{3.6}{x} + 0.02)dx$

$100 \ln|y| + 0.5y = 3.6 \ln|x| + 0.02x + C$

Since $y(10) = 10$,

$100 \ln 10 + 5 = 3.6 \ln 10 + 0.2 + C$

$C = 96.4 \ln 10 + 4.8$

Therefore $100 \ln|y| + 0.5y = 3.6 \ln|x| + 0.02x + 96.4 \ln 10 + 4.8$

(c)

$$
\begin{aligned}
\frac{dy}{dx} &= (1 + \ln x)y \\
\frac{1}{y}dy &= (1 + \ln x)dx \\
\int \frac{1}{y}dy &= \int (1 + \ln x)dx = \int dx + \int \ln x dx \\
\ln|y| &= x + x\ln x - x + C \\
\ln|y| &= x \ln x + C \\
|y| &= e^{x \ln x + C} = e^C e^{x \ln x} = e^C e^{\ln x^x} \\
|y| &= e^C x^x \ , \ e^C \text{ is a positive arbitrary constant}
\end{aligned}
$$

This gives the solution $y = kx^x$, k is an arbitrary constant.

Since $y(1) = 1$, we have $1 = k1^1$ so $k = 1$ and a solution is given by $y = x^x$.

13. Making the substitution $v(x) = u'(x)$ we have

$$\begin{aligned}
v'(x) &= -\frac{v(x)b}{x} \\
\frac{dv}{dx} &= -\frac{vb}{x} \\
\frac{1}{v}dv &= -\frac{b}{x}dx \\
\int \frac{1}{v}dv &= \int -\frac{b}{x}dx \\
\ln|v| &= -b(\ln|x| + C) \quad \text{where } C \text{ is an arbitrary constant.}
\end{aligned}$$

Since $v(x) = u'(x) > 0$ and $x > 0$, we have

$$\begin{aligned}
\ln v &= -b(\ln x + C) \\
v &= e^{-b\ln x + c} = e^{-b\ln x}e^{C} \\
u'(x) &= e^{-b\ln x}e^{C} \\
&= e^{\ln x^{-b}}e^{C} \\
&= e^{C}x^{-b} \\
&= kx^{-b} \text{ where } k = e^{c} \text{ is an arbitrary positive constant.}
\end{aligned}$$

Finally,

$$u(x) = \int kx^{-b}dx = \begin{cases} k\ln x + kk_2 & \text{if } b = 1 \\ \dfrac{kx^{1-b}}{1-b} + kk_2 & \text{if } b \neq 1 \end{cases}$$

where k_2 is another arbitrary constant. More simply, we may write

$$u(x) = \begin{cases} k_1 \ln(x) + k_2 & \text{if } b = 1 \\ k_1 x^{1-b} + k_2 & \text{if } b \neq 1 \end{cases}$$

where k_1, k_2 are arbitrary constants and $k_1 > 0$.

14. (a) Separating variables, $\frac{1}{B}dB = r\,dt$
Integrating,

$$\begin{aligned}
\int \frac{1}{B}dB &= \int r\,dt \\
\ln|B| &= rt + C \;,\; C \text{ is an arbitrary constant.}
\end{aligned}$$

We know $B > 0$, so

$$\begin{aligned}
\ln B &= rt + C \\
B &= e^{rt+C} \\
B &= e^{C}e^{rt}
\end{aligned}$$

When $t = 0$, $B = P$ so $P = e^C e^0 = e^c$.

Therefore the solution is

$$B = Pe^{rt} \; .$$

(b) We have $\dfrac{dB}{dt} = r(B - \dfrac{N}{r})$

$$\frac{1}{B - \frac{N}{r}} dB = r dt$$

Integrating,

$$\int \frac{1}{B - \frac{N}{r}} dB = \int r dt$$

$$\ln |B - \frac{N}{r}| = rt + C \; , \; C \text{ is an arbitrary constant}$$

$$B - \frac{N}{r} = e^{rt+C}$$

$$B = e^C e^{rt} + \frac{N}{r}$$

Since $B = P$ when $t = 0$,

$$P = e^C e^0 + \frac{N}{r}$$

$$P = e^C + \frac{N}{r}$$

$$e^C = P - \frac{N}{r}$$

Therefore the solution is $B = (P - \dfrac{N}{r})e^{rt} + \dfrac{N}{r}$.

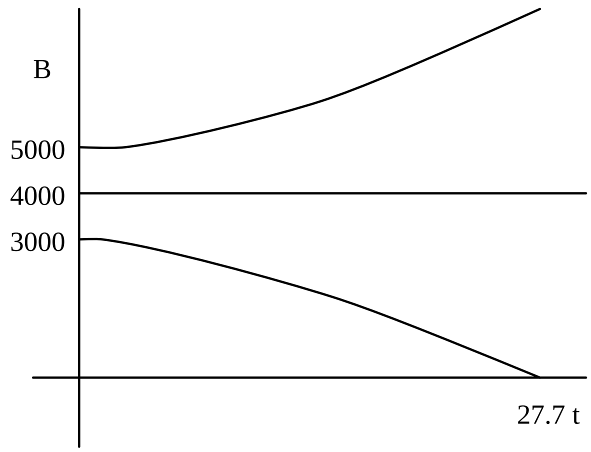
(c) With $r = 0.05$, $N = 200$, the balance equation becomes $B = (P - 4000)e^{0.05t} + 4000$.
With $P = 3000, 4000, 5000$ the balance equations are

$$B = -1000e^{0.05t} + 4000$$

$$B = 4000$$

$$B = 1000e^{0.05t} + 4000$$

with graphs:



15. (a) Separating variables,

$$\frac{1}{B}dB = (0.25t + 0.50)dt$$
$$\int \frac{1}{B} = \int (0.25t + 0.50)dt$$
$$\ln|B| = \frac{0.25t^2}{2} + 0.5t + C\ , \ C \text{ is an arbitrary constant.}$$

Since $B > 0$,

$$\ln B = 0.125t^2 + 0.5t + C$$
$$B = e^{0.125t^2+0.5t+C}$$
$$B = e^C e^{0.125t^2+0.5t}$$

When $t = 0$, $B = 100000$ so

$$100000 = e^C e^0 = e^C$$

so the solution is

$$B = 100000e^{0.125t^2+0.5t}$$

(b) When $t = 5$,

$$B = 100000e^{0.125(25)+0.5(5)}$$
$$= 100000e^{5.625}$$
$$= \$27727228.45$$

16. (a) Separating variables,

$$\frac{1}{(p-p_0)}dp = kdt$$
$$\int \frac{1}{(p-p_0)}dp = \int kdt$$
$$\ln|p-p_0| = kt + C$$

C is an arbitrary constant. $|p - p_0| = e^{kt+C}$.

(Case 1)

$$\begin{aligned} p &> p_0 \\ p - p_0 &= e^{kt+C} \\ p &= e^C e^{kt} + p_0 \\ &= C_2 e^{kt} + p_0 \end{aligned}$$

where $C_2 > 0$ is an arbitrary constant.

(Case 2)

$$\begin{aligned} p &< p_0 \\ p_0 - p &= e^{kt+C} \\ p &= p_0 - C_2 e^{kt} \end{aligned}$$

where $C_2 > 0$ is an arbitrary constant.

(b) Case 1: $\lim_{t\to\infty} p = \lim_{t\to\infty} C_2 e^{kt} + p_0 = p_0$ since $k < 0$
Case 2: $\lim_{t\to\infty} p = \lim_{t\to\infty} p_0 - C_2 e^{kt} = p_0$ since $k < 0$
in both cases, $p \to p_0$ as $t \to \infty$.

(c) Case 1: When $t = 0, p = C_2 e^0 + p_0 = C_2 + p_0$.
Case 2: When $t = 0, p = p_0 - C_2 e^0 = p_0 - C_2$. Thus when $t = 0$, the price is \$ C_2 above or below the equilibrium price.

17. (a)

$$\begin{aligned} \Gamma(1) &= \int_0^\infty e^{-t} dt \;=\; \lim_{a\to\infty} \int_0^a e^{-1} dt \;=\; \lim_{a\to\infty} -e^{-t}\Big|_0^a \\ &= \lim_{a\to\infty} -e^{-a} + 1 \\ &= 1 \end{aligned}$$

(b)

$$\begin{aligned} \Gamma(n+1) &= \int_0^\infty t^n e^{-t} dt \;=\; \lim_{a\to 0} \int_0^a t^n e^{-t} dt \\ & \quad u = t^n \quad dv = e^{-t} dt \\ & \quad du = nt^{n-1} \quad v = -e^{-t} \\ &= \lim_{a\to\infty} \left[-t^n e^{-t}\Big|_0^a + \int_0^a nt^{n-1} e^{-t} dt \right] \\ &= \lim_{a\to\infty} \left[-a^n e^{-a} + n \int_0^a t^{n-1} e^{-t} dt \right] \\ &= \lim_{a\to\infty} n \int_0^a t^{n-1} e^{-t} dt = n \int_0^\infty t^{n-1} e^{-t} dt \\ &= n\Gamma(n) \end{aligned}$$

(c) Given any positive integer n,

$$\left.\begin{aligned}\Gamma(n) &= (n-1)\Gamma(n-1)\\ &= (n-1)(n-2)\Gamma(n-2)\\ &= \dots\\ &= (n-1)(n-2)\dots(3)(2)(1)\Gamma(1)\end{aligned}\right\}\text{from (b)}$$
$$\begin{aligned} &= (n-1)(n-2)\dots(2) \quad \text{from (a)}\\ &= (n-1)!\end{aligned}$$

18. (a) Let $f(x) = \dfrac{2x}{\sqrt{x^2-1}}, g(x) = \dfrac{x^2}{\sqrt{x^2-1}}$

$$\begin{aligned}\int_2^\infty f(x)dx &= \int_2^\infty \frac{2x}{\sqrt{x^2-1}}dx = \lim_{a\to\infty}\int_2^a \frac{2x}{\sqrt{x^2-1}}dx\\ &= \lim_{a\to\infty}\int_{x=2}^{x=a}\frac{1}{\sqrt{u}}du \qquad \text{where } \begin{aligned}u &= x^2-1\\ du &= 2xdx\end{aligned}\\ &= \lim_{a\to\infty} 2\sqrt{u}\Big|_{x=2}^{x=a}\\ &= \lim_{a\to\infty} 2\sqrt{x^2-1}\Big|_2^a\\ &= \lim_{a\to\infty} 2\sqrt{a^2-1}-2\sqrt{3}\\ &= \infty\end{aligned}$$

Since $g(x) \geq f(x)$ for $x \geq 2$, $\int_2^\infty g(x)dx$ is also divergent.

(b) Let $f(x) = \dfrac{-2x}{(x^2+5)^2}, g(x) = \dfrac{\sqrt{-x}}{(x^2+5)^2}$

$$\begin{aligned}\int_{-\infty}^{-2} f(x)dx &= \int_{-\infty}^{-2}\frac{-2x}{(x^2+5)^2}dx\\ &= \lim_{a\to-\infty}\int_{-2}^{a}\frac{2x}{(x^2+5)^2}dx\\ &= \lim_{a\to-\infty}\int_{x=-2}^{x=a}\frac{1}{u^2}du \qquad \text{where } \begin{aligned}u &= x^2+5\\ du &= 2xdx\end{aligned}\\ &= \lim_{a\to-\infty} -\frac{1}{u}\Big|_{x=-2}^{x=a}\\ &= \lim_{a\to-\infty} -\frac{1}{x^2+5}\Big|_{-2}^{a}\\ &= \lim_{a\to-\infty}\left(\frac{1}{9}-\frac{1}{a^2+5}\right)\\ &= \frac{1}{9}\end{aligned}$$

so $\int_{-\infty}^{-2} f(x)dx$ is convergent.

Since $|f(x)| \geq |g(x)|$ for all $x \leq -2$, $\int_{-\infty}^{-2} g(x)dx$ is also convergent.

19. If $p = 0$, then

$$\int_1^\infty x^p dx = \int_1^\infty dx = \lim_{a\to\infty} \int_1^a dx = \lim_{a\to\infty} x\Big|_1^a = \lim_{a\to\infty} (a-1) = \infty$$

so it is divergent.

If $p = -1$, then

$$\begin{aligned}\int_1^\infty x^p dx &= \int_1^\infty \frac{1}{x} dx \\ &= \lim_{a\to\infty} \int_1^a \frac{1}{x} dx = \lim_{a\to\infty} \ln|x|\Big|_1^a = \lim_{a\to\infty} (\ln(a-1)) = \infty\end{aligned}$$

so it is divergent.

If $p \neq 0, -1$ then

$$\int_1^\infty x^p dx = \lim_{a\to\infty} \int_1^a x^p dx = \lim_{a\to\infty} \frac{x^{p+1}}{p+1}\Big|_1^a = \lim_{a\to\infty} \left(\frac{a^{p+1}}{p+1} - \frac{1}{p+1}\right)$$

For the limit to be finite, it is required $p + 1 \leq 0$, or $p \leq -1$. We know $p \neq -1$, so we know the integral is convergent if $p < -1$. If this is the case, then it has the value $-\frac{1}{p+1}$ given by the above limit.

20. (a)

$$\begin{aligned} r'(t) &= 1000t(-0.5)e^{-0.5t} + 1000e^{-0.5t} \\ &= 1000e^{-0.5t}(1 - 0.5t) \end{aligned}$$

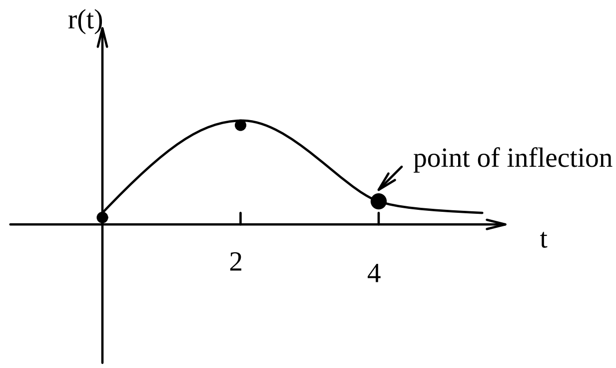
The only critical point is $t = 2$.

$$\begin{aligned} r''(t) &= 500e^{-0.5t}(0.5t - 1) - 500e^{-0.5t} \\ &= 500e^{-0.5t}(0.5t - 2) \end{aligned}$$

$t = 4$ is an inflection point where the concavity changes from negative to positive. Since $r''(2) < 0$, the critical point $t = 2$ is where an absolute maximum occurs.

$r(0) = 0$.

There are no vertical asymptotes, and a horizontal asymptote is at $r = 0$. Therefore the sketch is



(b) The graph shows an absolute maximum occurs when $t = 2$, so 2 days after the epidemic began people were getting sick the fastest.

(c) The total number of people who get sick is the number of people per day for each and every day as $t \to \infty$. Therefore, this is given by

$$\begin{aligned}\int_0^\infty r(t)dt &= \int_0^\infty 1000te^{-0.5t}dt \\ &= \lim_{y\to\infty} \int_0^y 1000te^{-0.5t}dt \\ &= \lim_{y\to\infty} -2000te^{-0.5t}\Big|_0^y + \int_0^y 2000e^{-0.5t}dt \\ &\quad \text{integrating by parts with} \\ &\quad \begin{array}{ll} u = 1000t & dv = e^{-0.5t} \\ du = 1000 & v = \frac{-e^{-0.5t}}{0.5} \\ & \quad = -2e^{-0.5t} \end{array} \\ &= \lim_{y\to\infty} -2000ye^{-0.5y} + 0 - (4000e^{-0.5t}\Big|_0^y) \\ &= \lim_{y\to\infty} -2000ye^{-0.5y} - 4000e^{-0.5y} + 4000 \\ &= 4000\end{aligned}$$

4000 people get sick altogether.

21. (a) We search for the profit function when the new version is bought at a time T. This is the revenues less the cost.

Revenues: Before time T, \$ R per unit of time was generated. Keeping in mind that inflation is r compounded continuously, the present value of \$ R at any time t is Re^{-rt}. So the total present value of the revenue generated by the early version is $\int_0^T Re^{-rt}dt$.

Similarly, after time T and continuing indefinitely thereafter (by assumption), the new version will generate a present value revenue of $\int_T^\infty Se^{-rt}dt$. So the present value of the revenue is $\int_0^T Re^{-rt}dt + \int_T^\infty Se^{-rt}dt$.

Cost: At time T, the new version will cost Ce^{-wT}. So the present value of the price of the new version is $Ce^{-wT}e^{-rT} = Ce^{-(r+w)T}$. This is the only cost.
Hence the profit function to be maximized is:

$$\begin{aligned}\pi(T) \;=\; \text{Revenue - Cost} \;&=\; \int_0^T Re^{-rt}dt + \int_T^\infty Se^{-rt}dt - Ce^{-(r+w)T}\\ &= \int_0^T Re^{-rt}dt + \lim_{b\to\infty}\left(\int_T^b Se^{-rt}dt\right) - Ce^{-(r+w)T}.\end{aligned}$$

And by the Fundamental Theorem of Calculus, this is:

$$= \;[F(T) - F(0)] + \lim_{b\to\infty}[G(b) - G(T)] - Ce^{-(r+w)T},$$

(where $\dfrac{dF(x)}{dx} = Re^{-rx}$, and $\dfrac{dG(x)}{dx} = Se^{-rx}$)

$$= F(T) - F(0) + [\lim_{b\to\infty} G(b)] - G(T) - Ce^{-(r+w)T}\;.$$

We now look for critical values by finding $\frac{d}{dT}\pi(T)$ and setting to zero:

$$\frac{d\pi(T)}{dT} \;=\; \frac{d}{dT}(F(T) - F(0) + \lim_{b\to\infty} G(b) - G(T) - Ce^{-(r+w)T})$$

and noticing that $F(0)$ and $\lim_{b\to\infty} G(b)$ are constants with respect to T, we get:

$$\begin{aligned}&= \; Re^{-rT} - Se^{-rT} + (r+w)Ce^{-(r+w)T}\\ &= \; (R-S)e^{-rT} + (r+w)Ce^{-(r+w)T}\end{aligned}$$

Setting this to zero, we solve for T^*:

$$\begin{aligned}(S-R)e^{-rT^*} \;&=\; (r+w)Ce^{-(r+w)T^*} \qquad \text{multiply both sides by } e^{rT^*}:\\ \Rightarrow\; S-R \;&=\; (r+w)Ce^{-wT^*}\\ \Rightarrow\; e^{-wT^*} \;&=\; \frac{S-R}{C(r+w)}\\ \Rightarrow\; -wT^* \;&=\; \ln\left(\frac{S-R}{C(r+w)}\right)\\ \Rightarrow\; T^* \;&=\; \frac{\ln\left(\frac{C(r+w)}{S-R}\right)}{w}\end{aligned}$$

Note: if T^* is zero or negative, then upgrade immediately! Also: The Fundamental Theorem of Calculus, strictly speaking, does not need to be used in solving this problem.

However, using it allows us to bypass performing an unnecessary integration and also an unnecessary differentiation.

(b) It is best to measure time in months. To this extent, monthly inflation is 1% (compounded continuously), since $e^{.12(\frac{1}{12})} = e^{r_m(1)} \Rightarrow\ r_m = 0.1$.

$R = 600$ and $S = 700$. To find C and w, note (if we set $t = 0$ to be the beginning of 1996), that:

$$Ce^{-w(0)} = 5,000 \Rightarrow C = 5,000.$$

Then, for the beginning of Feb. '96:

$$Ce^{-w(1)} = 4,901 \Rightarrow e^{-w} = \frac{4901}{5000} \Rightarrow w = -\ln\frac{4901}{5000} \approx .02.$$

So, from our results in part (a), $T^* = \dfrac{\ln\left(\frac{5000(.01+.02)}{700-600}\right)}{.02} = \dfrac{\ln\left(\frac{150}{100}\right)}{.02} \approx \dfrac{.4055}{.02} = 20.275$. So she should buy the Cat on Sept. 8th, 1997.