

Solutions to Supplementary Questions for HP Chapter 14

1. (a)

i) $\frac{d}{dx}(\ln(1+x) - \ln(2+x)) = \frac{1}{1+x} - \frac{1}{2+x}$

ii) $\frac{d}{dx}e^{x^2} = e^{x^2} \left(\frac{d}{dx}(x^2)\right) = 2xe^{x^2}$

iii) $\frac{d}{dx}(1+x^2)^{15} = 15(1+x^2)^{14} \left(\frac{d}{dx}(1+x^2)\right) = 30x(1+x^2)^{14}$

(b)

i) Since $\frac{d}{dx}e^{x^2} = 2xe^{x^2}$, then $\frac{d}{dx}e^{\frac{x^2}{2}} = xe^{\frac{x^2}{2}}$, hence $\int xe^{\frac{x^2}{2}} dx = e^{\frac{x^2}{2}} + c$

ii) The answer in (iii) is 30 times larger than the integrand, so try $\frac{1}{30}(1+x^2)^{15}$.

Here $\frac{d}{dx} \frac{1}{30}(1+x^2)^{15} = x(1+x^2)^{14}$, hence $\int x(1+x^2) dx = \frac{1}{30}(1+x^2)^{15}$.

iii) Let A, B be constants such that

$$\begin{aligned} \frac{A}{1+x} + \frac{B}{2+x} &= \frac{1}{(1+x)(2+x)} \Rightarrow \frac{(2+x)A + (1+x)B}{(1+x)(2+x)} = \frac{(A+B)x + (2A+B)}{(1+x)(2+x)} \\ &= \frac{1}{(1+x)(2+x)}. \end{aligned}$$

It now follows that:

(1) $A+B=0$ (since the coefficient of x is zero on the R.H.S.) (2) $2A+B=1$

Solving, we get $A=1, B=-1$, and so $\frac{1}{(1+x)(2+x)} = \frac{1}{1+x} - \frac{1}{2+x}$.

Hence $\int \frac{1}{(1+x)(2+x)} dx = \int \left(\frac{1}{1+x} - \frac{1}{2+x}\right) dx$, and from part (a) i), we see that this is $\ln(1+x) - \ln(2+x) + c$.

iv) Noting a pattern from question (ii), we try $(1+x^2)^{13}$: $\frac{d}{dx}(1+x^2)^{13} = 26x(1+x^2)^{12}$. This is 26 times too big, so: $\int x(1+x^2)^{12} dx = \frac{1}{26}(1+x^2)^{13}$.

v) Using the same strategy as (iii), set

$$\begin{aligned} \frac{1}{(1+x)(3+x)} &= \frac{A}{1+x} + \frac{B}{3+x} \\ &= \frac{(3+x)A + (1+x)B}{(1+x)(3+x)} \\ &= \frac{(A+B)x + (3A+B)}{(1+x)(3+x)}, \end{aligned}$$

so:

- (1) $A + B = 0$ (2) $3A + B = 1$.
 Solving, we get $A = \frac{1}{2}$ and $B = -\frac{1}{2}$ so

$$\begin{aligned} \int \frac{1}{(1+x)(3+x)} dx &= \int \left(\frac{1/2}{(1+x)} - \frac{1/2}{(3+x)} \right) dx \\ &= \frac{1}{2} \int \frac{1}{1+x} dx - \frac{1}{2} \int \frac{1}{3+x} dx \\ &= \frac{1}{2} \ln(1+x) - \frac{1}{2} \ln(3+x) + c \\ &= \ln \sqrt{1+x} + \ln \frac{1}{\sqrt{3+x}} + c \\ &= \ln \sqrt{\frac{1+x}{3+x}} + c \end{aligned}$$

2.

$$\begin{aligned} x^7 + x^6 - x - 1 &= (x+1)(x^6 - 1) \\ &= (x+1)(x-1)(x^5 + x^4 + x^3 + x^2 + x + 1) \\ &= (x^2 - 1)(x^5 + x^4 + x^3 + x^2 + x + 1) \end{aligned}$$

Also, $x^4 - x^2 = x^2(x^2 - 1)$ so

$$\begin{aligned} \int \frac{x^7 + x^6 - x - 1}{x^4 - x^2} dx &= \int \frac{(x^2 - 1)(x^5 + x^4 + x^3 + x^2 + x + 1)}{(x^2 - 1)x^2} dx \\ &= \int \frac{x^5}{x^2} dx + \int \frac{x^4}{x^2} dx + \int \frac{x^3}{x^2} dx + \int \frac{x^2}{x^2} dx + \int \frac{x}{x^2} dx + \int \frac{1}{x^2} dx \\ &= \int x^3 dx + \int x^2 dx + \int x dx + \int dx + \int \frac{1}{x} dx + \int \frac{1}{x^2} dx \\ &= \frac{x^4}{4} + \frac{x^3}{3} + \frac{x^2}{2} + x + \ln|x| - \frac{1}{x} + c \end{aligned}$$

3. First, we calculate $v(t)$ and $p(t)$:

1) $v(t) = \int a(t) dt = 6\left(\frac{1}{2}t^2\right) - 9t + c = 3t^2 - 9t + c$. But $v(1) = 0$, so $3(1)^2 - 9(1) + c = 0 \Rightarrow c = 6$, so $v(t) = 3t^2 - 9t + 6$ ($t \geq 0$)

2) $p(t) = \int v(t) dt = 3\left(\frac{1}{3}t^3\right) - 9\left(\frac{1}{2}t^2\right) + 6t + c' = t^3 - \frac{9}{2}t^2 + 6t + c'$, where $p(0) = 1$, and so $(0)^3 - \frac{9}{2}(0)^2 + 6(0) + c' = 1 \Rightarrow c' = 1$, so $p(t) = t^3 - \frac{9}{2}t^2 + 6t + 1$ ($t \geq 0$)

Now we answer the questions:

(a) $v(5) = 3(5)^2 - 9(5) + 6 = 75 - 45 + 6 = 36\text{m/s}$.

(b) $p(2) = 2^3 - \frac{9}{2}(2)^2 + 6(2) + 1 = 8 - 18 + 12 + 1 = 3\text{m}.$

(c) $v\left(\frac{5}{4}\right) = 3\left(\frac{5}{4}\right)^2 - 9\left(\frac{5}{4}\right) + 6 = \frac{75}{16} - \frac{180}{16} + \frac{96}{16} = -\frac{9}{16} \text{ m/s}$

$a\left(\frac{5}{4}\right) = 6\left(\frac{5}{4}\right) - 9 = \frac{15}{2} - \frac{18}{2} = -\frac{3}{2} \text{ m/s}^2$

The object has negative velocity at this time, and it is getting even more negative, so the object is speeding up.

(d) We check for critical points:

$$\begin{aligned} v(t) = 0 &\Rightarrow 3t^2 - 9(t) + 6 = 0 \\ &\Rightarrow t^2 - 3t + 2 = 0 \\ &\Rightarrow (t - 2)(t - 1) = 0 \\ &\Rightarrow t = 1 \text{ or } t = 2 \end{aligned}$$

1) Now, $p(1) = (1)^3 - \frac{9}{2}(1)^2 + 6(1) + 1 = \frac{7}{2}\text{m}.$, and this is a maximum, since $a(1) = 6(1) - 9 = -3 \text{ m/s}^2$ is negative.

2) $p(2) = 3\text{m}.$ (from (b) above), and this is a minimum, since $a(2) = 6(2) - 9 = 3 \text{ m/s}^2$ is positive.

We also check endpoints, so $p(0) = 1\text{m}.$, and $\lim_{t \rightarrow \infty} t^3 - \frac{9}{2}t^2 + 6t + 1 = \lim_{t \rightarrow \infty} t^3 = \infty.$

So the object starts at $x = 1\text{m}.$, moves right to $x = \frac{7}{2}\text{m}.$, then left to $x = 3\text{m}.$, and then moves right from then on. So the closest the object is to the origin is at the beginning, when $x = 1\text{m}.$

4.

$$\frac{df}{dx} = \frac{1}{x^2} \Rightarrow f(x) = \begin{cases} -\frac{1}{x} + c_1 & x > 0 \\ -\frac{1}{x} + c_2 & x < 0 \end{cases}$$

Now $f(1) = 1$, and since $x > 0$, we have $-\frac{1}{1} + c_1 = 1 \Rightarrow c_1 = 2$. But also $f(-1) = 2$, and since $x < 0$, we have $\frac{-1}{-1} + c_2 = 2 \Rightarrow c_2 = 1$. So

$$f(x) = \begin{cases} -\frac{1}{x} + 2 & x > 0 \\ -\frac{1}{x} + 1 & x < 0 \end{cases},$$

and $\frac{d}{dx}f(x) = \frac{1}{x^2} \quad (x \neq 0).$

5. (a) Since $\frac{d}{dx}e^{2x} = 2e^{2x}$, then $\frac{d}{dx} \frac{e^{2x}}{2} = e^{2x}$, hence $\int e^{2x} = \frac{e^{2x}}{2} + c$.

(b) Similarly, $\int 2^x dx = \frac{2^x}{\ln 2} + c$ since $\frac{d}{dx} \frac{2^x}{\ln 2} = \frac{\ln 2}{\ln 2} 2^x$.

(c) Since $\frac{d}{dx}a^{bx} = b \ln a(a^{bx})$, then $\int a^{bx} = \frac{a^{bx}}{b \ln a} + c$.

6. Let $u = 1 + x^2$. Then $du = 2x dx$, so $x dx = \frac{du}{2}$. Also, $x^2 = u - 1$, so $x^4 = (u - 1)^2$:

$$\begin{aligned} \int \sqrt{1+x^2} x^5 dx &= \int \sqrt{1+x^2} x^4 \cdot x dx \\ &= \int \sqrt{u}(u-1)^2 \frac{du}{2} \\ &= \frac{1}{2} \int \sqrt{u}(u^2 - 2u + 1) du \\ &= \frac{1}{2} \int (u^{5/2} - 2u^{3/2} + u^{1/2}) du \\ &= \frac{1}{2} \left(\frac{2}{7} u^{7/2} - 2 \frac{2}{5} u^{5/2} + \frac{2}{3} u^{3/2} \right) + c \\ &= \frac{1}{7} (1+x^2)^{7/2} - \frac{2}{5} (1+x^2)^{5/2} + \frac{1}{3} (1+x^2)^{3/2} + c \end{aligned}$$

7. (a)

$$\begin{aligned} \frac{d}{dx} \left(\frac{x^2}{2(1-x^2)} + c \right) &= \frac{2x(2(1-x^2)) - (-4x)(x^2)}{[2(1-x^2)]^2} \\ &= \frac{4x - 4x^3 + 4x^3}{4(1-x^2)^2} \\ &= \frac{x}{(1-x^2)^2} . \end{aligned}$$

So $\int \frac{x}{(1-x^2)^2} dx = \frac{x^2}{2(1-x^2)} + c$.

(b) Let $u = 1 - x^2$. Then $du = -2x dx$. So

$$\begin{aligned} \int \frac{x}{(1-x^2)^2} dx &= -\frac{1}{2} \int \frac{-2x}{(1-x^2)^2} dx \\ &= -\frac{1}{2} \int \frac{1}{u^2} du \\ &= -\frac{1}{2} \left(-\frac{1}{u} \right) + c \\ &= \frac{1}{2u} + c \\ &= \frac{1}{2(1-x^2)} + c \end{aligned}$$

Although in both parts (a) and (b), we write “ c ” for the constant of integration, this is sloppy and we should write (for example) C_1 for part (a) and C_2 for part (b). Now,

$$\left(\frac{x^2}{2(1-x^2)} + C_1\right) - \left(\frac{1}{2(1-x^2)} + C_2\right) = \frac{x^2 - 1}{2(1-x^2)} + C_1 - C_2 = -\frac{1}{2} + C_1 - C_2,$$

and so the two competing solutions differ by a constant. In fact, if we let $C_2 = C_1 - \frac{1}{2}$, then $-\frac{1}{2} + C_1 - C_2 = 0$, and hence $\left(\frac{x^2}{2(1-x^2)} + C_1\right) = \left(\frac{1}{2(1-x^2)} + C_2\right)$.

8.

$$\begin{aligned} \int [f(x)g''(x) - g(x)f''(x)]dx &= \int [f(x)g''(x) + f'(x)g'(x)] - [g(x)f''(x) + g'(x)f'(x)] \\ &= \int [f(x)g''(x) + f'(x)g'(x)]dx - \int [g(x)f''(x) + g'(x)f'(x)]dx \\ &= f(x)g'(x) + C_1 - [g(x)f'(x) + C_2] \quad (\text{Let } C_3 = C_1 - C_2) \\ &= f(x)g'(x) - g(x)f'(x) + C_3 \end{aligned}$$

9.

SOLUTION A.

$$\int \frac{e^x - 1}{e^x + 1} dx = \int \frac{e^x}{e^x + 1} dx - \int \frac{1}{e^x + 1} dx$$

i) $\int \frac{e^x}{e^x + 1} dx$: Let $u = e^x + 1$. Then $du = e^x dx$, and

$$\int \frac{e^x}{e^x + 1} dx = \int \frac{1}{u} = \ln |u| + C_1 = \ln |e^x + 1| + C_1 = \ln(e^x + 1) + C_1$$

ii) $\int \frac{1}{e^x + 1} dx$. Multiply through by $\frac{e^{-x}}{e^{-x}}$: $\int \frac{1}{e^x + 1} dx = \int \frac{e^{-x}}{1 + e^{-x}} dx$.

Now let $u = 1 + e^{-x}$. Then $du = -e^{-x} dx$, so

$$\begin{aligned} \int \frac{e^{-x}}{1 + e^{-x}} dx &= - \int \frac{-e^{-x}}{1 + e^{-x}} dx = - \int \frac{1}{u} du = - \ln |u| + C_2 = - \ln |1 + e^{-x}| + C_2 \\ &= - \ln(1 + e^{-x}) + C_2 \end{aligned}$$

$$\begin{aligned} \therefore \int \frac{e^x - 1}{e^x + 1} dx &= (\ln(e^x + 1) + C_1) - (- \ln(1 + e^{-x}) + C_2) \quad (\text{Let } C_3 = C_1 - C_2) \\ &= \ln((e^x + 1)(e^{-x} + 1)) + C_3 \\ &= \ln(2 + e^x + e^{-x}) + C_3 \end{aligned}$$

OR alternatively,

SOLUTION B.

Multiply through by $\frac{(e^{-x} + 1)}{(e^{-x} + 1)}$ to get $\int \frac{(e^x - 1)(e^{-x} + 1)}{(e^x + 1)(e^{-x} + 1)} dx = \int \frac{e^x - e^{-x}}{2 + e^x + e^{-x}} dx$.
Now let $u = 2 + e^x + e^{-x}$. Then $du = (e^x - e^{-x})dx$, and so:

$$\begin{aligned}\int \frac{e^x - 1}{e^x + 1} dx &= \int \frac{e^x - e^{-x}}{2 + e^x + e^{-x}} dx \\ &= \int \frac{1}{u} du \\ &= \ln |u| + C \\ &= \ln |2 + e^x + e^{-x}| + C \\ &= \ln(2 + e^x + e^{-x}) + C\end{aligned}$$

10.

$$\begin{aligned}C'(x) = 4,000 + \frac{(2x - 1)(5x + 3)}{x - 5} &\Rightarrow C(x) = \int (4,000 + \frac{(2x - 1)(5x + 3)}{x - 5}) dx \\ &= 4,000x + C_1 + \int \frac{(2x - 1)(5x + 3)}{x - 5} dx\end{aligned}$$

Now,

$$\begin{aligned}\int \frac{(2x - 1)(5x + 3)}{x - 5} dx &= \int \frac{10x^2 + x - 3}{x - 5} dx \\ &= \int ((10x + 51) + \frac{252}{x - 5}) dx \\ &= \int 10x dx + \int 51 dx + \int (\frac{252}{x - 5}) dx \\ &= 5x^2 + 51x + 252 \ln |x - 5| + C_2\end{aligned}$$

$$\begin{aligned}\therefore C(x) &= 4,000x + 5x^2 + 51x + 252 \ln |x - 5| + K \quad (\text{where } K = C_1 + C_2) \\ &= 5x^2 + 4,051x + 252 \ln |x - 5| + K\end{aligned}$$

Since the fixed costs are 1,000,000 Riyal, $C(0) = 1,000,000$.

Hence $5(0)^2 + 4,051(0) + 252 \ln |0 - 5| + C = 1,000,000$ Riyal,

$$\Rightarrow 252 \ln 5 + C = 1,000,000$$

$$\Rightarrow C = 1,000,000 - 252 \ln 5 \quad (\ln 5 \approx 1.609)$$

$$\approx 999,594 \text{ Riyal}$$

So $C(x) = x^2 + 4,051x + 252 \ln |x - 5| + 999,594$ Riyal.

11. (a)

$$\begin{aligned}\sum_{i=1}^n (2i - 1) &= \sum_{i=1}^n 2i + \sum_{i=1}^n -1 \\ &= 2 \sum_{i=1}^n i - \sum_{i=1}^n 1 \\ &= 2 \left(\frac{n(n+1)}{2} \right) - n \\ &= n(n+1) - n \\ &= n^2\end{aligned}$$

(b)

$$\begin{aligned}\sum_{i=1}^n (2i - 1)^2 &= \sum_{i=1}^n (4i^2 - 4i + 1) \\ &= 4 \sum_{i=1}^n i^2 - 4 \sum_{i=1}^n i + \sum_{i=1}^n 1 \\ &= 4 \left(\frac{n(n+1)(2n+1)}{6} \right) - 4 \left(\frac{n(n+1)}{2} \right) + n \\ &= \frac{2n(n+1)(2n+1) - 6n(n+1) + 3n}{3} \\ &= \frac{2n(n+1)(2n+1) - 3n((2n+2) - 1)}{3} \\ &= \frac{2n(2n+1)(n+1) - 3n(2n+1)}{3} \\ &= \frac{n(2n+1)[(2n+2) - 3]}{3} \\ &= \frac{n(2n+1)(2n-1)}{3}\end{aligned}$$

12.

SOLUTION 1. As was stated, the length of a side of the square is:

$$1 + 2 + 3 + \cdots + n = \sum_{i=1}^n i = \frac{n(n+1)}{2},$$

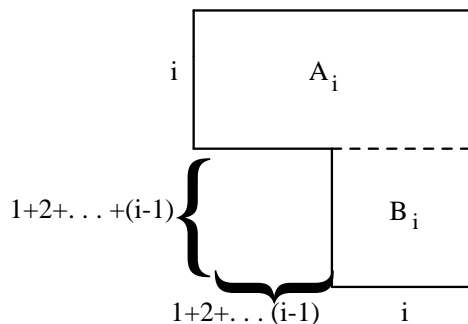
and hence the area of the square is $\left[\frac{n(n+1)}{2} \right]^2$.

That is one way to calculate the area of the square.

SOLUTION 2. Another way to calculate the area of the square is to find the area of each gnomon G_i , and then sum these areas.

So consider an arbitrary gnomon $G_i (1 \leq i \leq n)$. (In particular the following *does* work for $i = 1$ as well).

G_i looks as follows:



Divide the gnomon into two pieces A_i and B_i by extending the “middle” line.

The area of A_i is $i\left(\frac{i(i+1)}{2}\right)$.

The area of B_i is $i(1+2+3+\dots+(i-1))$.

$$= i\left(\frac{(i-1)((i-1)+1)}{2}\right)$$

$$= i\left(\frac{(i-1)i}{2}\right) \text{ (note if } i = 1 \text{ then this is zero as it should be)}$$

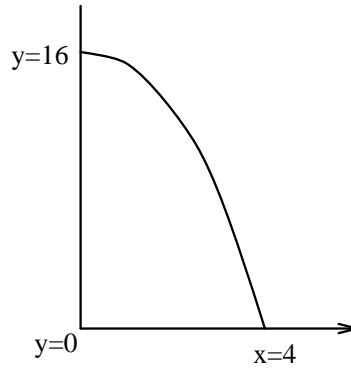
So the area of G_i is:

$$\begin{aligned} G_i = A_i + B_i &= i\left(\frac{i(i+1)}{2}\right) + i\left(\frac{i(i-1)}{2}\right) \\ &= \frac{i^2}{2}((i+1) + (i-1)) \\ &= \frac{i^2}{2}(2i) = i^3 \end{aligned}$$

So the area of the square, which is the sum of the areas of the gnomons, is $\sum_{i=1}^n i^3$.

We conclude that $\sum_{i=1}^n i^3 = \left[\frac{n(n+1)}{2}\right]^2$ because Solution 1 and Solution 2 both give the area of the same square.

- 13.** (a) In short, $f'(x) = -2x$, $f''(x) = -2$ is concave down everywhere. Also $f'(x) = 0 \Rightarrow x = 0$, and lastly $f(0) = 16$ and $f(4) = 0$. We sketch:



(b)

$$\begin{aligned} \text{i) } \bar{S}_4 &= f(1) + f(2) + f(3) + f(4) \\ &= 15 + 12 + 7 + 0 \\ &= 34 \end{aligned}$$

$$\begin{aligned} \text{ii) } \underline{S}_4 &= f(0) + f(1) + f(2) + f(3) \\ &= 16 + 15 + 12 + 7 \\ &= 50 \end{aligned}$$

$$\begin{aligned} \text{iii) } S_4 &= f\left(\frac{1}{2}\right) + f\left(\frac{3}{2}\right) + f\left(\frac{5}{2}\right) + f\left(\frac{7}{2}\right) \\ &= \frac{63}{4} + \frac{55}{4} + \frac{39}{4} + \frac{15}{4} = \frac{172}{4} \\ &= 43 \end{aligned}$$

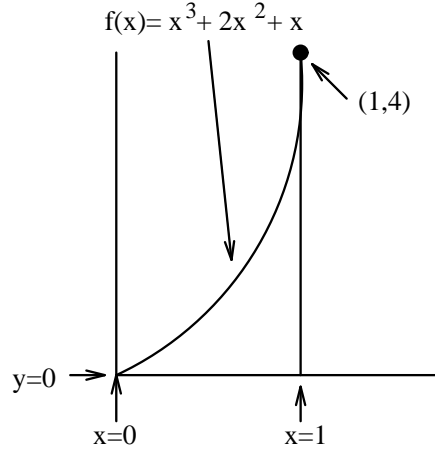
(c)

$$\begin{aligned} \lim_{n \rightarrow \infty} \bar{S}_n &= \lim_{n \rightarrow \infty} \left[\frac{4}{n} \left(16 - \left(\frac{4}{n} \right)^2 \right) + \frac{4}{n} \left(16 - \left(\frac{8}{n} \right)^2 \right) + \cdots + \frac{4}{n} \left(16 - \left(\frac{4n}{n} \right)^2 \right) \right] \\ &= \lim_{n \rightarrow \infty} \left[\frac{4}{n} \left(16n - \frac{4^2}{n^2} \left(\sum_{i=1}^n i^2 \right) \right) \right] \\ &= \lim_{n \rightarrow \infty} \left[64 - \frac{64}{n^3} \left(\frac{n(n+1)(2n+1)}{6} \right) \right] \\ &= 64 - \lim_{n \rightarrow \infty} \frac{32}{3} \left(1 + \frac{1}{n} \right) \left(2 + \frac{1}{n} \right) \\ &= 64 - \frac{64}{3} \\ &= \frac{128}{3} \\ &= 42\frac{2}{3} \end{aligned}$$

14. (a) The only problematic curve is $f(x) = x^3 + 2x^2 + x$.
In short, $f'(x) = 3x^2 + 4x + 1$ and $f''(x) = 6x + 4$.

Letting $f'(x) = 0$, we get $3x^2 + 4x + 1 = 0 \Rightarrow x = -1$ or $x = -\frac{1}{3}$. There is an inflection point at $x = -\frac{2}{3}$, but we are only concerned with x in $[0, 1]$, so everywhere in $[0, 1]$, the curve is concave up (since $f''(0) = 6(0) + 4 = 4$, for instance).

Noting that $f(0) = 0$ and $f(1) = 1 + 2 + 1 = 4$, we graph:



(b) First, we calculate S_n : Since the distance from $x = 1$ to $x = 0$ is one, the length of each subinterval is just $\frac{1}{n}$, so:

$$\begin{aligned}
 S_n &= \frac{1}{n}f\left(\frac{1}{n}\right) + \frac{1}{n}f\left(2\left(\frac{1}{n}\right)\right) + \frac{1}{n}f\left(3\left(\frac{1}{n}\right)\right) + \cdots + \frac{1}{n}\left(f\left(n\left(\frac{1}{n}\right)\right)\right) \\
 &= \frac{1}{n}\left(f\left(\frac{1}{n}\right) + f\left(\frac{2}{n}\right) + \cdots + f\left(\frac{n}{n}\right)\right) \\
 &= \frac{1}{n}\left(\left[\left(\frac{1}{n}\right)^3 + 2\left(\frac{1}{n}\right)^2 + \left(\frac{1}{n}\right)\right] + \left[\left(\frac{2}{n}\right)^3 + 2\left(\frac{2}{n}\right)^2 + \left(\frac{2}{n}\right)\right] + \cdots + \left[\left(\frac{n}{n}\right)^3 + 2\left(\frac{n}{n}\right)^2 + \left(\frac{n}{n}\right)\right]\right) \\
 &= \frac{1}{n}\left(\left[\left(\frac{1}{n}\right)^3 + \left(\frac{2}{n}\right)^3 + \cdots + \left(\frac{n}{n}\right)^3\right] + \left[2\left(\frac{1}{n}\right)^2 + 2\left(\frac{2}{n}\right)^2 + \cdots + 2\left(\frac{n}{n}\right)^2\right] \right. \\
 &\quad \left. + \left[\left(\frac{1}{n}\right) + \left(\frac{2}{n}\right) + \cdots + \left(\frac{n}{n}\right)\right]\right) \\
 &= \frac{1}{n}\left(\frac{1}{n^3}\left(\sum_{i=1}^n i^3\right) + \frac{2}{n^2}\left(\sum_{i=1}^n i^2\right) + \frac{1}{n}\left(\sum_{i=1}^n i\right)\right) \\
 &= \frac{1}{n}\left(\frac{1}{n^3}\left(\frac{n(n+1)}{2}\right)^2 + \frac{2}{n^2}\left(\frac{n(n+1)(2n+1)}{6}\right) + \frac{1}{n}\left(\frac{n(n+1)}{2}\right)\right) \\
 &= \frac{1}{4}\frac{n^2(n+1)^2}{n^4} + \frac{1}{3}\frac{n(n+1)(2n+1)}{n^3} + \frac{1}{2}\frac{n(n+1)}{n^2}
 \end{aligned}$$

$$\begin{aligned}
 \text{Now, } \lim_{n \rightarrow \infty} S_n &= \lim_{n \rightarrow \infty} \left(\frac{1}{4}\left(1 + \frac{1}{n}\right)^2 + \frac{1}{3}\left(1 + \frac{1}{n}\right)\left(2 + \frac{1}{n}\right) + \frac{1}{2}\left(1 + \frac{1}{n}\right)\right) \\
 &= \frac{1}{4} + \frac{2}{3} + \frac{1}{2} = \frac{17}{12}
 \end{aligned}$$

This is the area of the region.

15. (a) Using the hint, we guess that there is an interval of length one, each subinterval being of length $\frac{1}{n}$. By doing this, we get $\frac{i^4}{n^4}$ remaining within the \sum sign, which is leading us towards $f(x) = x^4$.

$$\lim_{n \rightarrow \infty} \sum_{i=1}^n \frac{i^4}{n^5} = \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{i=1}^n \frac{i^4}{n^4}.$$

$$\begin{aligned} \text{Hence, } S_n &= \frac{1}{n} \sum_{i=1}^n \left(\frac{i}{n}\right)^4 \\ &= \frac{1}{n} \left[\left(\frac{1}{n}\right)^4 + \left(\frac{2}{n}\right)^4 + \left(\frac{3}{n}\right)^4 + \cdots + \left(\frac{n}{n}\right)^4 \right] \end{aligned}$$

Letting $f(x) = x^4$ as per the hint, we get:

$$= \frac{1}{n} f\left(\frac{1}{n}\right) + \frac{1}{n} f\left(\frac{2}{n}\right) + \frac{1}{n} f\left(\frac{3}{n}\right) + \cdots + \frac{1}{n} f\left(\frac{n}{n}\right)$$

This is just summing the areas of the n rectangles found under the function $f(x)$ from $x = 0$ to $x = 1$.

$$\text{Hence, } \lim_{n \rightarrow \infty} \left(\sum_{i=1}^n \frac{i^4}{n^5} \right) = \int_0^1 x^4 dx$$

$$\begin{aligned} \text{(b) } S_n &= \frac{1}{n} \sum_{i=1}^n \frac{1}{1 + \left(\frac{i}{n}\right)^2}. \text{ Let } f(x) = \frac{1}{1 + x^2} \\ \Rightarrow S_n &= \frac{1}{n} f\left(\frac{1}{n}\right) + \frac{1}{n} f\left(\frac{2}{n}\right) + \cdots + \frac{1}{n} f\left(\frac{n}{n}\right) \\ \Rightarrow \lim_{n \rightarrow \infty} S_n &= \int_0^1 \frac{1}{1 + x^2} dx \end{aligned}$$

$$\begin{aligned} \text{(c) } S_n &= \sum_{i=1}^n \left[3\left(1 + \frac{2i}{n}\right)^5 - 6 \right] \frac{2}{n} \\ &= \frac{2}{n} \sum_{i=1}^n \left[3\left(1 + \frac{2i}{n}\right)^5 - 6 \right]. \text{ Let } f(x) = 3(1 + x)^5 - 6 \\ &= \frac{2}{n} \sum_{i=1}^n f\left(\frac{2i}{n}\right) \\ &= \frac{2}{n} f\left(\frac{2}{n}\right) + \frac{2}{n} f\left(\frac{4}{n}\right) + \frac{2}{n} f\left(\frac{6}{n}\right) + \cdots + \frac{2}{n} f\left(\frac{2n}{n}\right) \\ \text{Hence, } \lim_{n \rightarrow \infty} S_n &= \int_0^2 (3(1 + x)^5 - 6) dx. \end{aligned}$$

16. First, $\int_{-a}^a x^2 dx = \frac{x^3}{3} \Big|_{-a}^a = \frac{a^3}{3} - \frac{(-a)^3}{3} = \frac{2a^3}{3}$. Now,

$$\text{(a) } \int_0^{2a} (u + a)^2 du = \frac{(u + a)^3}{3} \Big|_0^{2a} = \frac{(3a)^3}{3} - \frac{a^3}{3} = \frac{26a^3}{3}$$

$$(b) \quad 2 \int_0^a x^2 dx = 2 \left(\frac{x^3}{3} \Big|_0^a \right) = 2 \left(\frac{a^3}{3} - 0 \right) = \frac{2a^3}{3}$$

$$(c) \quad - \int_a^{-a} x^2 dx = \int_{-a}^a x^2 dx \left(= \frac{2a^3}{3} \right)$$

$$(d) \quad \int_{-a}^a (x^2 + x) dx = \left(\frac{x^3}{3} + \frac{x^2}{2} \right) \Big|_{-a}^a = \left(\frac{a^3}{3} + \frac{a^2}{2} \right) - \left(\frac{(-a)^3}{3} - \frac{(-a)^2}{2} \right) \\ = \left(\frac{a^3}{3} - \frac{-a^3}{3} \right) + \left(\frac{a^2}{2} - \frac{a^2}{2} \right) = \frac{2a^3}{3}$$

$$(e) \quad \int_0^{2a} (u-a)^2 du = \frac{(u-a)^3}{3} \Big|_0^{2a} = \frac{(2a-a)^3}{3} - \frac{(0-a)^3}{3} = \frac{a^3}{3} - \frac{-a^3}{3} = \frac{2a^3}{3}.$$

So the answer is (a).

17. First, calculate $\int_{g(x)}^{h(x)} f(t) dt$:

By the Fundamental Theorem, this is

$$F(h(x)) - F(g(x)), \quad \text{where } F \text{ is any antiderivative of } f.$$

Now,

$$\begin{aligned} \frac{d}{dx} \left(\int_{g(x)}^{h(x)} f(t) dt \right) &= \frac{d}{dx} (F(h(x)) - F(g(x))) \\ &= \frac{d}{dx} F(h(x)) - \frac{d}{dx} F(g(x)) \\ &= f(h(x))h'(x) - f(g(x))g'(x) \blacksquare \end{aligned}$$

18. (a)

$$x^2 - 1 \geq 0 \Leftrightarrow (x-1)(x+1) \geq 0$$

$$\Leftrightarrow (x-1) = 0, x+1 = 0, [(x-1) > 0 \text{ and } (x+1) > 0] \text{ or } [(x-1) < 0 \\ \text{and } (x+1) < 0]$$

$$\Leftrightarrow x = 1, x = -1, (x > 1 \text{ and } x > -1) \text{ or } (x < 1 \text{ and } x < -1)$$

$$\Leftrightarrow x \geq 1 \text{ or } x \leq -1$$

Then $x^2 - 1 < 0 \Leftrightarrow -1 < x < 1$.

Now

$$\begin{aligned}
 \int_{-2}^3 |x^2 - 1| dx &= \int_{-2}^{-1} |x^2 - 1| dx + \int_{-1}^1 |x^2 - 1| dx + \int_1^3 |x^2 - 1| dx \\
 &= \int_{-2}^{-1} (x^2 - 1) dx + \int_{-1}^1 -(x^2 - 1) dx + \int_1^3 (x^2 - 1) dx \\
 &= \left(\frac{x^3}{3} - x \right) \Big|_{-2}^{-1} + \left(-\frac{x^3}{3} + x \right) \Big|_{-1}^1 + \left(\frac{x^3}{3} - x \right) \Big|_1^3 \\
 &= \left(\left[\frac{(-1)^3}{3} - (-1) \right] - \left[\frac{(-2)^3}{3} - (-2) \right] \right) + \left(\left[\frac{-(-1)^3}{3} + 1 \right] - \left[\frac{-(-1)^3}{3} + (-1) \right] \right) \\
 &\quad + \left(\left[\frac{(3)^3}{3} - (3) \right] - \left[\frac{(1)^3}{3} - (1) \right] \right) \\
 &= \left[\frac{2}{3} - \left(-\frac{2}{3}\right) \right] + \left[\frac{2}{3} - \left(-\frac{2}{3}\right) \right] + \left[6 - \left(-\frac{2}{3}\right) \right] \\
 &= \frac{4}{3} + \frac{4}{3} + \frac{20}{3} = \frac{28}{3}
 \end{aligned}$$

(b)

$$\begin{aligned}
 \int_{-1}^2 (x - 2|x|) dx &= \int_{-1}^0 (x - 2|x|) dx + \int_0^2 (x - 2|x|) dx \\
 &= \int_{-1}^0 (x - 2(-x)) dx + \int_0^2 (x - 2x) dx \\
 &= \int_{-1}^0 3x dx + \int_0^2 -x dx \\
 &= \frac{3}{2} x^2 \Big|_{-1}^0 - \frac{x^2}{2} \Big|_0^2 \\
 &= \left(0 - \frac{3}{2}(-1)^2 \right) - \left(\frac{2^2}{2} - 0 \right) \\
 &= -\frac{3}{2} - \frac{4}{2} = -\frac{7}{2}
 \end{aligned}$$

(c) First, $x - 1 \geq 0 \Leftrightarrow x \geq 1$. Now

$$\begin{aligned}
 \int_0^2 (x^2 - |x - 1|) dx &= \int_0^1 (x^2 - |x - 1|) dx + \int_1^2 (x^2 - |x - 1|) dx \\
 &= \int_0^1 (x^2 - (-(x - 1))) dx + \int_1^2 (x^2 - (x - 1)) dx \\
 &= \int_0^1 (x^2 + x - 1) dx + \int_1^2 (x^2 - x + 1) dx \\
 &= \left(\frac{x^3}{3} + \frac{x^2}{2} - x \right) \Big|_0^1 + \left(\frac{x^3}{3} - \frac{x^2}{2} + x \right) \Big|_1^2 \\
 &= \left(-\frac{1}{6} - 0 \right) + \left(\frac{16}{6} - \frac{5}{6} \right) = \frac{5}{3}
 \end{aligned}$$

19. (a) Although we could use the strategy suggested above, since we know $\int \frac{1}{x} dx$, we compute directly: $\int \frac{1}{x} dx = \ln|x| + C$.

i) $F(2) = 0 \Rightarrow \ln|2| + C = 0 \Rightarrow C = -\ln|2|$, so $F(x) = \ln|x| - \ln|2| = \ln|\frac{x}{2}|$.

ii) $F(2) = -3$. Here, we just need a function that is (uniformly) three less than the answer to (a), so we obtain: $F(x) = \ln|\frac{x}{2}| - 3$.

(b) Since we presently do not know how to obtain $\int \sqrt{1+x^3} dx$, we use the strategy suggested above. By the Preamble, $\int_a^x (\sqrt{1+t^3}) dt$ is an antiderivative of $\sqrt{1+x^3}$ for any constant a .

i) $F(3) = 0$. Now we must choose an appropriate a . We need: $\int_a^3 (\sqrt{1+t^3}) dt = 0$.

Now it becomes obvious that we can choose $a = 3$, and now $F(x) = \int_3^x (\sqrt{1+t^3}) dt$.

ii) $F(3) = 1$. This function is just (uniformly) one more than the function obtained in part (a): $F(x) = (\int_3^x \sqrt{1+t^3} dt) + 1$.

20. (a) $\int_0^2 4x^3 dx = x^4 \Big|_0^2 = 16$

For Simpson's rule, $n = 4$ so $h = \frac{2-0}{4} = \frac{1}{2}$ and

$$\begin{aligned} \int_0^2 F(x) dx &\approx \frac{1}{6} [4(4)(\frac{1}{2})^3 + 2(4)(1)^3 + 4(4)(\frac{3}{2})^3 + (4)(2)^3] \\ &= \frac{1}{6}(4) [\frac{1}{2} + 2 + \frac{27}{2} + 8] \\ &= \frac{2}{3} [\frac{48}{2}] \\ &= 16, \text{ so the error is zero.} \end{aligned}$$

(b) Utilizing the same strategy as question 22 of Chapter 16, we find $\int_{-h}^h f(x) dx$ for the cubic polynomial $ax^3 + bx^2 + cx + d$. Let $g_1(x) = ax^3$ and let $g_2(x) = bx^2 + cx + d$.

Now,

$$\begin{aligned}
 \int_{-h}^h f(x)dx &= \int_{-h}^h g_1(x)dx + \int_{-h}^h g_2(x)dx \\
 &= \int_{-h}^h ax^3dx + \frac{h}{3}[g_2(-h) + 4g_2(0) + g_2(h)] \\
 &\quad \text{(this is from the results of question 22, since } g_2 \text{ is a quadratic function)} \\
 &= \frac{ax^4}{4} \Big|_{-h}^h + \frac{h}{3}[g_2(-h) + 4g_2(0) + g_2(h)] \\
 &= \frac{ah^4}{4} - \frac{a(-h)^4}{4} + \frac{h}{3}[g_2(-h) + 4g_2(0) + g_2(h) + (a(-h)^3 + 4a(0)^3 + a(h)^3)] \\
 &= 0 + \frac{h}{3}[(g_1(-h) + g_2(-h)) + (4g_1(0) + 4g_2(0)) + (g_1(h) + g_2(h))] \\
 &= \frac{h}{3}[f(-h) + 4f(0) + f(h)]
 \end{aligned}$$

The rest of the proof is exactly the same as that of question 22.

21. (a)

$$\begin{aligned}
 T_n &= \frac{h}{2}[f(a) + 2f(a+h) + 2f(a+2h) + \dots + 2f(a+(n-1)h) + f(b)] \\
 4T_n &= h[2f(a) + 4f(a+h) + 4f(a+2h) + \dots + 4f(a+(n-1)h) + 2f(b)]
 \end{aligned}$$

Using the same notation for $T_{n/2}$, with half the number of subintervals each interval will have twice the length, so we have

$$T_{n/2} = h[f(a) + 2f(a+2h) + 2f(a+4h) + \dots + 2f(a+(n-1)2h) + f(b)]$$

so

$$\begin{aligned}
 4T_n - T_{n/2} &= h[2f(a) - f(a) + 4f(a+h) + (4f(a+2h) - 2f(a+2h)) + \dots + 4f(a+(n-1)h) \\
 &\quad + (2f(b) - f(b))] \\
 &= h[f(a) + 4f(a+h) + 2f(a+2h) + \dots + 4f(a+(n-1)h) + f(b)]
 \end{aligned}$$

$$\begin{aligned}
 \frac{1}{3}(4T_n - T_{n/2}) &= \frac{h}{3}[f(a) + 4f(a+h) + 2f(a+2h) + \dots + 4f(a+(n-1)h) + f(b)] \\
 &= S_n
 \end{aligned}$$

$$(b) \int_0^2 e^x dx = e^x \Big|_0^2 = e^2 - 1 \approx 6.389056099$$

$$T_2 = \frac{1}{2}(e^0 + 2e^1 + e^2) \approx 6.912809878$$

$$T_4 = \frac{1}{2}(e^0 + 2e^{1/2} + 2e + 2e^{3/2} + e^2) \approx 6.52161011$$

$$S_4 = \frac{1}{3}(4T_4 - T_2) \approx \frac{1}{3}(4(6.52161011) - 6.912809878)$$

$$\approx 6.391210187$$

The errors are:

$$T_2 : -0.523753779$$

$$T_4 : -0.132554011$$

$$S_4 : -0.002154088$$

Simpson's rule with 4 subintervals is most accurate.

22. $\int_0^1 x^{5/2} dx = \frac{2}{7} x^{7/2} \Big|_0^1 = \frac{2}{7}$ For each rule, $h = \frac{(b-a)}{n} = \frac{1}{4}$. For the corrected trapezoidal rule, $f'(x) = \frac{5}{2} x^{3/2}$

Trapezoidal rule: $\frac{1}{8} [2(\frac{1}{4})^{5/2} + 2(\frac{1}{2})^{5/2} + 2(\frac{3}{4})^{5/2} + 1^{5/2}]$

Corrected trap. rule: $\frac{1}{8} [2(\frac{1}{4})^{5/2} + 2(\frac{1}{2})^{5/2} + 2(\frac{3}{4})^{5/2} + 1^{5/2}] - \frac{1}{192} [\frac{5}{2} 1^{3/2}]$

Simpson's rule: $\frac{1}{12} [4(\frac{1}{4})^{5/2} + 2(\frac{1}{2})^{5/2} + 4(\frac{3}{4})^{5/2} + 1^{5/2}]$ The results are:

	<u>Area</u>	<u>Error</u>
Trapezoidal rule:	0.298791496	-0.013077211
Corrected trap. rule:	0.285770663	-0.000056377
Simpson's rule:	0.285592546	0.000121740

The Corrected trapezoidal rule provides the best approximation, with Simpson's rule second.

23. (a)

(i) For $f(x) = \sqrt{x}$, $f''(x) = -\frac{1}{4x^{3/2}}$

For x in the interval $(1, 4)$, $|f''(x)| < \frac{1}{4}$ which means $|E| \left| \frac{(b-a)^3}{12n^2} f''(c) \right| < \left| \frac{3^3}{12n^2} \frac{1}{4} \right|$.

We need $\left| \frac{3^3}{12n^2} \frac{1}{4} \right| < 0.01$ so $n^2 > \frac{3^3}{12 \cdot 4 \cdot 0.01} = 56.25$, $n > 7.5$ or $n > 7$.

(ii) Simpson's rule $f^{(3)}(x) = \frac{3}{8x^{5/2}}$ and $f^{(4)}(x) = -\frac{15}{16x^{7/2}}$

For x in the interval $(1, 4)$, $|f^{(4)}(x)| < \frac{15}{16}$ so $|E| = \left| \frac{(b-a)^4 f^{(4)}(c)}{180n^5} \right| < \frac{3^4}{180n^5} \frac{15}{16}$ for all c in the interval $(1, 4)$. We need $\frac{3^4}{180n^5} \frac{15}{16} < 0.01$ so

$$n^5 > \frac{3^4}{180} \frac{15}{16} \frac{1}{0.01} = 42.1875$$

$$n > 2.11 \text{ or just } n > 2.$$

23. (b)

(i) $f(x) = x^5$, $f'(x) = 5x^4$, $f''(x) = 20x^3$

For x in the interval $(1, 3)$, $|f''(x)| < 540$ so $|E| = \left| \frac{(b-a)^3 f''(c)}{12n^2} \right| < \frac{2^3}{12n^2} (540)$ for all c in the interval $(1, 3)$. Therefore we need $\frac{2^3}{12n^2} (540) < 0.01$,

$$n^2 > \frac{2^3 (540)}{12 \cdot 0.01} = 36000$$

$$n > 189.7 \text{ or just } n > 189.$$

(ii) $f^{(3)}(x) = 60x^2$, $f^{(4)}(x) = 120x$

For x in the interval $(1, 3)$, $|f^{(4)}(x)| < 360$

so $|E| = \left| \frac{(b-a)^4}{180n^5} f''(c) \right| < \frac{2^4}{180n^5} (360)$ for all c in the interval $(1, 3)$.

Therefore we need $\frac{2^4}{180n^5} (360) < 0.01$,

$$n^5 > \frac{2^4 (360)}{180(0.01)}$$

$$n^5 > 3200$$

$$n > 5$$

24. We find the points of intersection of $f(x)$ and the x -axis, i.e. we find the roots of $f(x)$:

$$f(x) = x^4 + x^3 - 4x^2 - 4x = x(x^3 + x^2 - 4x - 4).$$

Since the coefficients of x^3 and x^2 are identical, and likewise the coefficients of x and x^0 are identical, we try $(x+1)$:

$$(x^3 + x^2 - 4x - 4) = (x^2 - 4)(x + 1).$$

And since $(x^2 - 4) = (x - 2)(x + 2)$, we get :

$$f(x) = (x + 2)(x + 1)x(x - 2), \text{ so the roots are } x = -2, -1, 0, 2.$$

Now, if x is in $(-2, -1)$, then $f(x)$ is negative.

If x is in $(-1, 0)$, then $f(x)$ is positive.

If x is in $(0, 2)$, then $f(x)$ is negative.

So, we find:

$$\begin{aligned} & \int_{-2}^{-1} -(x^4 + x^3 - 4x^2 - 4x)dx + \int_{-1}^0 (x^4 + x^3 - 4x^2 - 4x)dx + \int_0^2 -(x^4 + x^3 - 4x^2 - 4x)dx \\ &= -\left(\frac{x^5}{5} + \frac{x^4}{4} - \frac{4}{3}x^3 - 2x^2\right)\Big|_{-2}^{-1} + \left(\frac{x^5}{5} + \frac{x^4}{4} - \frac{4}{3}x^3 - 2x^2\right)\Big|_{-1}^0 - \left(\frac{x^5}{5} + \frac{x^4}{4} - \frac{4}{3}x^3 - 2x^2\right)\Big|_0^2 \\ &= -\left[\left(-\frac{1}{5} + \frac{1}{4} + \frac{4}{3} - 2\right) - \left(-\frac{32}{5} + 4 + \frac{32}{3} - 8\right)\right] \\ &\quad + \left[0 - \left(-\frac{1}{5} + \frac{1}{4} + \frac{4}{3} - 2\right)\right] - \left[\left(\frac{32}{5} + 4 - \frac{32}{3} - 8\right) - 0\right] \\ &= -\left[-\frac{37}{60} - \frac{4}{15}\right] + \left[\frac{37}{60}\right] - \left[-\frac{124}{15}\right] \\ &= \frac{37}{60} + \frac{37}{60} + \frac{16}{60} + \frac{496}{60} = \frac{586}{60} = \frac{293}{30} = 9\frac{23}{30} \end{aligned}$$

25. We know that $\int_1^b f(x)dx = \sqrt{b^2 + 1} - \sqrt{2}$.

Remembering the Fundamental Theorem (which states that $\int_a^b f(x) = F(b) - F(a)$ where F is an antiderivative of f), then it is a “safe bet” that $F(b) = \sqrt{b^2 + 1}$ and $F(1) = \sqrt{2}$, since

$$\int_1^b f(x)dx = F(b) - F(1) = \sqrt{b^2 + 1} - \sqrt{2} \quad (\text{where } F'(x) = f(x)).$$

So a wise guess for $F(x)$ is $F(x) = \sqrt{x^2 + 1}$, where $F'(x) = f(x)$. So to find $f(x)$, we just calculate:

$$\frac{d}{dx}(\sqrt{x^2 + 1}) = \frac{d}{dx}(x^2 + 1)^{1/2} = \frac{2x}{2(x^2 + 1)^{1/2}} = \frac{x}{(x^2 + 1)^{1/2}}.$$

Now, we double-check:

$$\begin{aligned} \int_1^b \frac{x}{(x^2 + 1)^{1/2}} dx &= \int_1^b \frac{1}{2} \frac{2x}{(x^2 + 1)^{1/2}} dx = \int_*^* \frac{1}{2u^{1/2}} du = u^{1/2} \Big|_*^* \\ &= (x^2 + 1)^{1/2} \Big|_1^b = \sqrt{b^2 + 1} - \sqrt{2}. \checkmark \end{aligned}$$

So $f(x) = \frac{x}{\sqrt{x^2 + 1}}$.

26. (a) As per the hint, we translate everything m units to the left. This obviously does not change the area that we are looking for. Now, we find the area:

$$\begin{aligned}
 A &= \int_{-h}^h (px^2 + qx + r)dx \\
 &= \left(\frac{px^3}{3} + \frac{qx^2}{2} + rx \right) \Big|_{-h}^h \\
 &= \left(\frac{p(h)^3}{3} + \frac{q(h)^2}{2} + rh \right) - \left(\frac{p(-h)^3}{3} + \frac{q(-h)^2}{2} + r(-h) \right) \\
 &= \frac{2}{3}p(h)^3 + 2rh \\
 &= \frac{1}{3}h(2ph^2 + 6r) \\
 &= \frac{1}{3}h[(p(-h)^2 + r) + 4r + (p(h)^2 + r) - qh + qh] \\
 &= \frac{1}{3}h[(p(-h)^2 + q(-h) + r) + (4p(0)^2 + 4q(0) + 4r) + (p(h)^2 + q(h) + r)] \\
 &= \frac{1}{3}h[f(-h) + 4f(0) + f(h)]
 \end{aligned}$$

Substituting back, we get $\frac{1}{3}h[f(a) + 4f(m) + f(b)]$.

- (b) Since n is even, we can divide $[a, b]$ into the $\frac{n}{2}$ different intervals:

$$[a, a + 2h], [a + 2h, a + 4h], \dots, [a + 2(k-1)h, a + 2kh], \dots, [a + (n-4)h, a + (n-2)h], [a + (n-2)h, a + nh(=b)] .$$

We now apply part (a) to each interval. So for $1 \leq k \leq \frac{n}{2}$, we consider $[a + 2(k-1)h, a + 2kh]$: By part (a), if we let $h' = \frac{1}{2}((a + 2kh) - (a + 2(k-1)h)) = \frac{1}{2}(2h) = h$, and if we note $m = \frac{((a + 2(k-1)h) + (a + 2kh))}{2} = a + (2k-1)h$, then the area of f under this region is:

$$\frac{h'}{3}[f(a + (2k-2)h) + 4f(a + (2k-1)h) + f(a + 2kh)] .$$

Now summing all of these regions for $1 \leq k \leq \frac{n}{2}$, we get (since $h' = h$):

$$\begin{aligned}
 &\frac{h}{3}[f(a) + 4f(a+h) + f(a+2h)] + \frac{h}{3}[f(a+2h) \\
 &+ 4f(a+3h) + f(a+4h)] + \dots + \frac{h}{3}[f(a+(2k-2)h) + 4f(a+(2k-1)h) \\
 &+ f(a+2kh)] + \dots + \frac{h}{3}[f(a+(n-2)h) + 4f(a+(n-1)h) + f(b)] \\
 &= \frac{h}{3}[f(a) + 4f(a+h) + 2f(a+2h) + \dots + 4f(a+(n-1)h) + f(b)]
 \end{aligned}$$

27. (a) $\frac{dy}{dx} = \frac{3}{2}(x-1)^{1/2}$ so

$$L = \int_2^{10} \sqrt{1 + \frac{9}{4}(x-1)} dx = \int_2^{10} \sqrt{\frac{9}{4}x - \frac{5}{4}} dx.$$

Let $u = \frac{9}{4}x - \frac{5}{4}$, $du = \frac{9}{4}dx$

$$\begin{aligned} L &= \int_{x=2}^{x=10} \frac{4}{9} \sqrt{u} du \\ &= \frac{8}{27} u^{3/2} \Big|_{x=2}^{x=10} \\ &= \frac{8}{27} \left(\frac{9}{4}x - \frac{5}{4} \right)^{3/2} \Big|_{x=2}^{x=10} \\ &= \frac{8}{27} \left[\left(\frac{85}{4} \right)^{3/2} - \left(\frac{13}{4} \right)^{3/2} \right] \\ &= \frac{85^{3/2} - 13^{3/2}}{27} \end{aligned}$$

(b) $8x^2 = y^3$ so $y = 2x^{2/3}$ and $\frac{dy}{dx} = \frac{4}{3}x^{-1/3}$

$$\begin{aligned} L &= \int_1^{2\sqrt{2}} \sqrt{1 + \frac{16}{9}x^{-2/3}} dx = \int_1^{2\sqrt{2}} \sqrt{\frac{x^{2/3} + 16/9}{x^{2/3}}} dx \\ &= \int_1^{2\sqrt{2}} \frac{\sqrt{x^{2/3} + 16/9}}{x^{1/3}} dx \end{aligned}$$

Let

$$\begin{aligned} u &= x^{2/3} + \frac{16}{9} \\ du &= \frac{2}{3x^{1/3}} dx \\ \frac{3}{2} du &= \frac{1}{x^{1/3}} dx \end{aligned}$$

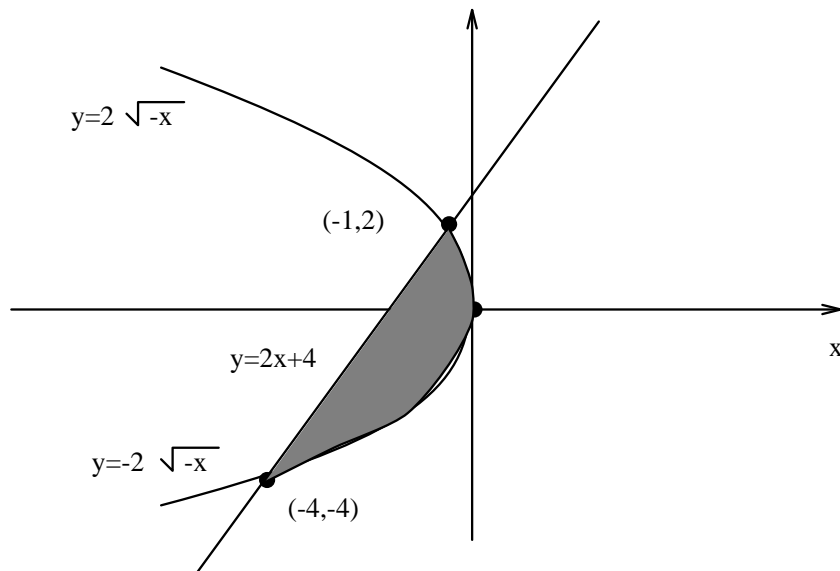
so

$$\begin{aligned} L &= \int_{x=1}^{x=2\sqrt{2}} \frac{3}{2} \sqrt{u} du = u^{3/2} \Big|_{x=1}^{x=2\sqrt{2}} \\ &= \left(x^{2/3} + \frac{16}{9} \right)^{3/2} \Big|_1^{2\sqrt{2}} = \left(x^{2/3} + \frac{16}{9} \right)^{3/2} \Big|_1^{2\sqrt{2}} \\ &= \left(2 + \frac{16}{9} \right)^{3/2} - \left(1 + \frac{16}{9} \right)^{3/2} = \left(\frac{34}{9} \right)^{3/2} - \left(\frac{25}{9} \right)^{3/2} \\ &= \frac{34\sqrt{34} - 125}{27} \end{aligned}$$

(c) $24xy = x^4 + 48$ so $y = \frac{x^4 + 48}{24x} = \frac{x^3}{24} + \frac{2}{x}$, $\frac{dy}{dx} = \frac{x^2}{8} - \frac{2}{x^2}$

$$\begin{aligned} L &= \int_2^3 \sqrt{1 + \left(\frac{x^2}{8} - \frac{2}{x^2}\right)^2} dx = \int_2^3 \sqrt{1 + \left(\frac{x^4}{64} - \frac{1}{2} + \frac{4}{x^4}\right)} dx \\ &= \int_2^3 \sqrt{\frac{1}{64x^4}(x^8 + 32x^4 + 256)} dx \\ &= \int_2^3 \frac{1}{8x^2} \sqrt{(x^4 + 16)^2} dx = \int_2^3 \frac{x^4 + 16}{8x^2} dx \\ &= \int_2^3 \frac{x^2}{8} + \frac{2}{x^2} dx = \left(\frac{x^3}{24} - \frac{2}{x}\right) \Big|_2^3 = \frac{27}{24} - \frac{2}{3} - \frac{8}{24} + 1 \\ &= \frac{27 - 16 - 8 + 24}{24} = \frac{27}{24} = \frac{9}{8} \end{aligned}$$

28. (a) The region bounded is shown in the graph:



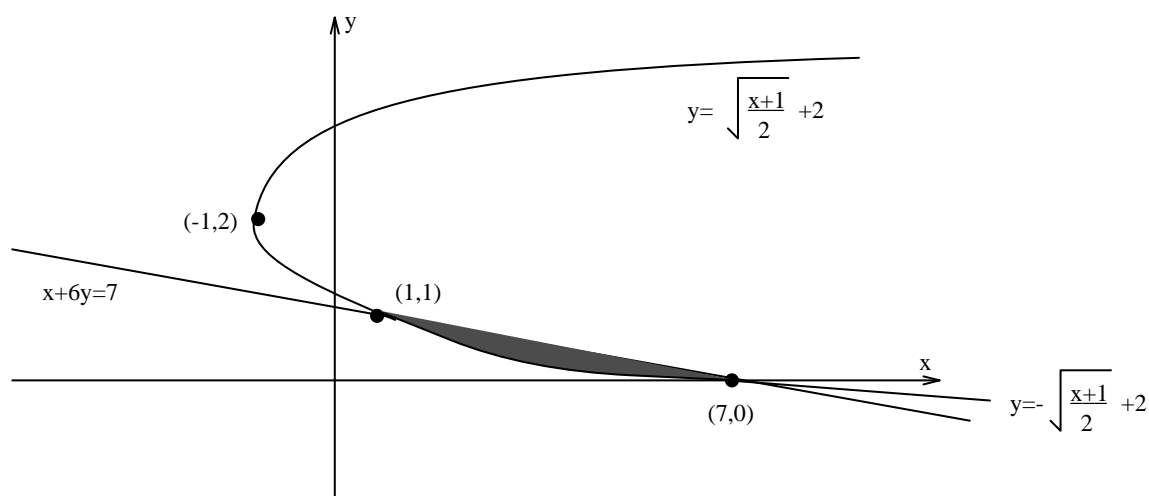
(i) Integrating with respect to x , the area is given by

$$\begin{aligned} &\int_{-4}^0 (2\sqrt{-x}) - (-2\sqrt{-x}) dx + \int_{-4}^{-1} (2x + 4) - (-2\sqrt{-x}) dx \\ &= \int_{-4}^0 4\sqrt{-x} dx + \int_{-4}^{-1} 2x + 4 + 2\sqrt{-x} dx \\ &= -\frac{8}{3}(-x)^{3/2} \Big|_{-4}^0 + x^2 + 4x - \frac{4}{3}(-x)^{3/2} \Big|_{-4}^{-1} \\ &= \frac{8}{3} + 1 - 4 - \frac{4}{3} - 16 + 16 + \frac{4}{3}(8) \\ &= \frac{36}{3} - 3 = 9 \end{aligned}$$

(ii) Integrating with respect to y , the area is

$$\begin{aligned} \int_{-4}^2 \left(\frac{-y^2}{4} - \left(\frac{y-4}{2} \right) \right) dy &= \frac{1}{2} \int_{-4}^2 -\frac{1}{2}y^2 - y + 4 dy \\ &= \frac{1}{2} \left[-\frac{1}{6}y^3 - \frac{1}{2}y^2 + 4y \right]_{-4}^2 \\ &= \frac{1}{2} \left[-\frac{4}{3} - 2 + 8 - \frac{32}{3} + 8 + 16 \right] = \frac{1}{2} \left(-\frac{36}{3} + 30 \right) \\ &= 9 \end{aligned}$$

(b) The region bounded is shown:



(i) Integrating with respect to x , the area is

$$\begin{aligned} \int_1^7 \left(-\frac{1}{6}x + \frac{7}{6} \right) - \left(-\sqrt{\frac{x+1}{2}} + 2 \right) dx &= \int_1^7 -\frac{1}{6}x - \frac{5}{6} + \sqrt{\frac{x+1}{2}} dx \\ &= -\frac{1}{12}x^2 - \frac{5}{6}x + \frac{4}{3} \left(\frac{x+1}{2} \right)^{3/2} \Big|_1^7 = \left(-\frac{49}{12} - \frac{35}{6} + \frac{32}{3} + \frac{1}{12} + \frac{5}{6} - \frac{4}{3} \right) \\ &= \frac{-49 - 70 + 128 + 1 + 10 - 16}{12} = \frac{4}{12} = \frac{1}{3} \end{aligned}$$

(ii) and with respect to y ,

$$\begin{aligned} \int_0^1 (7 - 6y) - (2(y-2)^2 - 1) dy &= \int_0^1 7 - 6y + 1 - 2y^2 + 8y - 8 dy \\ &= \int_0^1 -2y^2 + 2y dy = 2 \int_0^1 -y^2 + y dy = 2 \left[-\frac{1}{3}y^3 + \frac{1}{2}y^2 \right]_0^1 \\ &= 2 \left[-\frac{1}{3} + \frac{1}{2} \right] = 2 \left(\frac{1}{6} \right) = \frac{1}{3} \end{aligned}$$

29. The slope of $y = x + 2$ is one. So we find the point on $y = x^2$ with slope one:

$$y' = 2x, \quad y' = 1 \Rightarrow 2x = 1 \Rightarrow x = \frac{1}{2}, \quad \text{and the point is } C = \left(\frac{1}{2}, \frac{1}{4}\right).$$

We now determine the distance from C to the line: From basic mathematics, we take the line perpendicular to $y = x + 2$ that goes through C , and find out where it intersects with $y = x + 2$ (there are more sophisticated methods that are shorter, but this works).

The line we are looking for has slope $\frac{-1}{1} = -1$ and passes through $C = \left(\frac{1}{2}, \frac{1}{4}\right)$:

Using $y - y_1 = m(x - x_1)$, we get $y - \frac{1}{4} = -1\left(x - \frac{1}{2}\right) \Rightarrow y = -x + \frac{1}{2} + \frac{1}{4} = -x + \frac{3}{4}$

Now, we find the point of intersection between $y = x + 2$ and $y = -x + \frac{3}{4}$:

$$x + 2 = -x + \frac{3}{4} \Rightarrow 2x = -\frac{5}{4} \Rightarrow x = -\frac{5}{8}.$$

So the point of intersection P is $\left(-\frac{5}{8}, 1\frac{3}{8}\right)$. And the distance from P to C is

$$\sqrt{\left(\frac{1}{2} - \left(-\frac{5}{8}\right)\right)^2 + \left(\frac{1}{4} - 1\frac{3}{8}\right)^2} = \sqrt{\frac{81}{64} + \frac{81}{64}} = \frac{9\sqrt{2}}{8}.$$

Now we find A and B (the intersection points of $y = x^2$ and $y = x + 2$)

$$x^2 = x + 2 \Rightarrow x^2 - x - 2 = 0 \Rightarrow (x - 2)(x + 1) = 0 \Rightarrow x = 2, -1$$

So $A = (-1, 1)$ and $B = (2, 4)$. The distance from A to B is: $\sqrt{(4 - 1)^2 + (2 - (-1))^2} = 3\sqrt{2}$.

And now the area of triangle $ABC = \frac{1}{2}\overline{AB} \overline{PC} = \frac{1}{2}(3\sqrt{2})\left(\frac{9\sqrt{2}}{8}\right) = \frac{27}{8}$

The area between the line and parabola (integrate w.r.t. x):

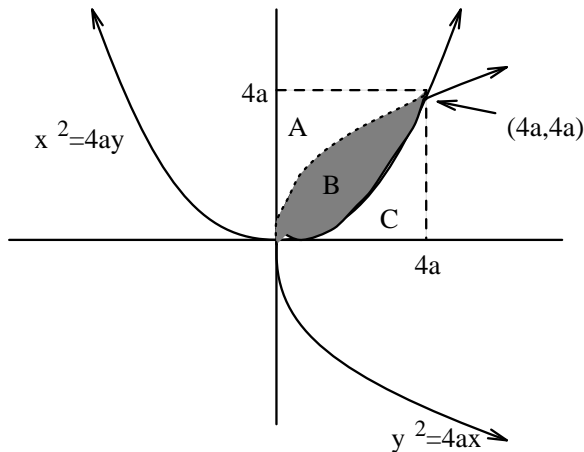
$$\begin{aligned} \int_{-1}^2 ((x + 2) - x^2) dx &= \int_{-1}^2 x dx + 2 \int_{-1}^2 dx - \int_{-1}^2 x^2 dx = \frac{x^2}{2} \Big|_{-1}^2 + 2(x) \Big|_{-1}^2 - \left(\frac{x^3}{3}\right) \Big|_{-1}^2 \\ &= \left(2 - \frac{1}{2}\right) + 2(2 - (-1)) - \left(\frac{8}{3} - \frac{-1}{3}\right) = \frac{3}{2} + 6 - 3 \\ &= \frac{9}{2} \\ &= \frac{4}{3} \left(\frac{27}{8}\right) = \frac{4}{3} \quad (\text{Area of triangle } ABC). \end{aligned}$$

30. Find the points of intersection:

$$\begin{aligned}
 x^2 = 4ay &\Rightarrow y = \frac{x^2}{4a} \\
 \Rightarrow y^2 = 4ax &\Rightarrow \left(\frac{x^2}{4a}\right)^2 = 4ax \\
 \Rightarrow x^4 - (4a)^3x &= 0 \\
 \Rightarrow x(x^3 - (4a)^3) &= 0 \\
 \Rightarrow x(x - 4a)(x^2 + 4ax + (4a)^2) &= 0
 \end{aligned}$$

But note that since $a > 0$, then $x^2 + 4ax + (4a)^2 \neq 0$ for any $x \in \mathbb{R}$, so $x = 0, 4a$.
 So the points of intersection are $(0, 0)$ and $(4a, 4a)$.

The area looks as follows:



Now, $x^2 - 4ay \Rightarrow y = \frac{x^2}{4a}$.
 Also, $y^2 = 4ax$
 $\Rightarrow y = +\sqrt{4ax}$
 (we need the upper half of the parabola)

We find the value of

$$\begin{aligned}
 \int_0^{4a} \left(\sqrt{4ax} - \frac{x^2}{4a}\right) dx &= 2\sqrt{a} \int_0^{4a} x^{1/2} dx - \frac{1}{4a} \int_0^{4a} x^2 dx \\
 &= 2\sqrt{a} \left(\frac{2}{3}x^{3/2}\right) \Big|_0^{4a} - \frac{1}{4a} \left(\frac{x^3}{3}\right) \Big|_0^{4a} \\
 &= \frac{4}{3}a^{1/2}(4a)^{3/2} - \frac{(4a)^3}{12a} \\
 &= \frac{32a^2}{3} - \frac{16a^2}{3} \\
 &= \frac{(4a)^2}{3}
 \end{aligned}$$

NOTE: A MUCH QUICKER METHOD:

We find the area of C:

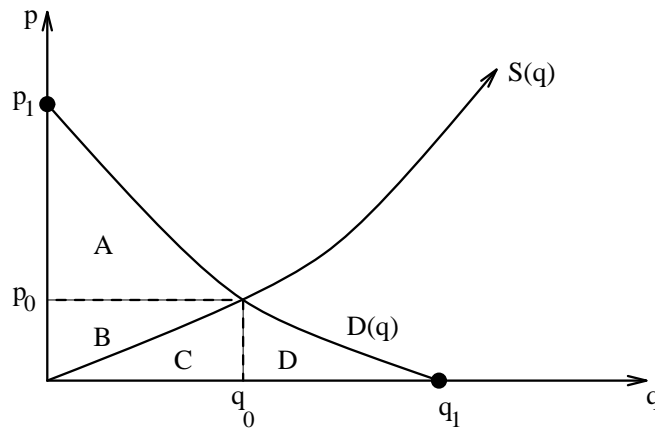
$$\int_0^{4a} \frac{x^2}{4a} dx = \frac{1}{4a} \left(\frac{x^3}{3}\right) \Big|_0^{4a} = \frac{(4a)^2}{3} .$$

By symmetry, the area of A is also $\frac{(4a)^2}{3}$. Now the area of B is the area of the $4a \times 4a$ square less the area of A and C

$$\begin{aligned} &= (4a)^2 - \left(\frac{(4a)^2}{3} + \frac{(4a)^2}{3}\right) \\ &= \frac{(4a)^2}{3} \end{aligned}$$

31. (a) p_1 : The maximum price that any consumer is willing to pay.

(b) q_1 : The maximum quantity that any consumer needs/wants.



(c) $\int_0^{q_0} S(q) dq$: This is region C .

It represents the accumulated revenue obtained by all producers who are willing to sell the product for a price of $\leq p_0$ if each of these producers sold their product at the minimum price that each was willing to sell it for.

(d) $\int_0^{q_0} D(q) dq$: This is regions A , B and C .

It represents the accumulated spending of all consumers who are willing to buy the product for a price of $\geq p_0$ if each of these consumers bought their product at the maximum price that each was willing to buy it for.

(e) $p_0 q_0$: This is regions B and C .

It represents the total revenue of the producers (who are willing to sell the product for a price of $\leq p_0$) if each of these producers sold their product at the equilibrium price p_0 .

(f) $\int_0^{q_0} (D(q) - S(q)) dq$: This is regions A and B .

This represents the total gain enjoyed by the system as a whole, for producers and consumers alike (who are willing to sell/buy for a price of $\leq / \geq p_0$). This is obvious since the region is the sum of the consumers' surplus and producers' surplus.