Supplementary Questions for HP Chapter 14

1. (a) Find derivatives for the following function
   i) $\ln(1+x) - \ln(2+x)$
   ii) e^{x^2}
   iii) $(1+x^2)^{15}$

   (b) Using the answers from part (a), find antiderivatives for the following functions:
   i) $xe^{\frac{x^2}{2}}$
   ii) $x(1+x^2)^{14}$
   iii) $\dfrac{1}{(1+x)(2+x)}$ (Hint: express this in terms of $\dfrac{A}{1+x} + \dfrac{B}{2+x}$: find A and B.)
   iv) $x(1+x^2)^{12}$
   v) $\dfrac{1}{(1+x)(3+x)}$

2. Find $\displaystyle\int \frac{x^7 + x^6 - x - 1}{x^4 - x^2}\,dx$.

3. If we are given the acceleration of an object $a(t)$ (in terms of time), then the velocity $v(t)$ of the object is $v(t) = \int a(t)dt$, and the position $p(t)$ of the object is: $p(t) = \int v(t)dt$.

   The acceleration of an object moving along the x-axis is given in metres per second squared by

   $$a(t) = 6t - 9$$

   where t is time in seconds. If the object has a velocity of zero after one second and if the object starts from position $x = 1$m. at time $t = 0$, answer each of the following questions assuming $t \geq 0$:

   (a) What is the velocity of the object at $t = 5$s.?
   (b) What is the position of the object at $t = 2$s.?
   (c) Is the object speeding up or slowing down at $t = \dfrac{5}{4}$s ?
   (d) What is the closest the object ever comes to the origin?

4. Find a function $y = f(x)$ satisfying $\dfrac{dy}{dx} = \dfrac{1}{x^2}$ and passes through the points $(1, 1)$ & $(-1, 2)$.
   (Hint: there may be two different values of the constant of integration, one for each side of the point where the function is undefined.)

5. Evaluate:

(a) $\displaystyle\int e^{2x}\,dx$

(b) $\displaystyle\int 2^x\,dx$

(c) $\displaystyle\int a^{bx}\,dx$

6. Find $\displaystyle\int \sqrt{1+x^2}\ x^5\,dx$

(Hint: Let $u = 1 + x^2$, and write x^5 as $x^4 \cdot x$.)

7. (a) Verify the formula by differentiation of the right-hand side.

$$\int \frac{x}{(1-x^2)^2}\,dx = \frac{x^2}{2(1-x^2)} + C$$

(b) Find $\displaystyle\int \frac{x}{(1-x^2)^2}\,dx$ by substition of u for $1 - x^2$. Is the answer consistent with the formula of part (a)? Explain.

8. Calculate $\displaystyle\int [f(x)g''(x) - g(x)f''(x)]dx$.

(Hint: Consider $f'(x)g'(x)$ and the product rule.)

9. Find $\displaystyle\int \frac{e^x - 1}{e^x + 1}\,dx$

10. Consider the costs of drilling an oil well. There are two types of costs: fixed costs (independent of drilling) and marginal costs (here, the increment in costs that comes from drilling one more meter). Together, they make up the total cost, C. The marginal costs depend on the depth at which you are drilling; drilling becomes more expensive, per meter, as you dig deeper into the earth. Suppose the fixed costs are 1,000,000 Riyal (the Riyal is the unit of currency of Saudi Arabia), and the marginal costs are

$$C'(x) = 4,000 + \frac{(2x-1)(5x+3)}{x-5}$$

in Riyal/meter, where x is the depth in meters. Find the total cost of drilling a well x meters deep. Note that since the fixed costs are 1,000,000 Riyal, this means that $C(0) = 1,000,000$.

11. Using Equations from HP, Chap. 16, derive compact formulas in terms of n for the following sums:

(a) $\displaystyle\sum_{i=1}^{n} (2i - 1)$

(b) $\displaystyle\sum_{i=1}^{n} (2i - 1)^2$

12. The following was published by Abu Bekr Mohammed ibn Alhusain Alkarchi in about A.D. 1010. The figure below shows a square $ABCD$ in which sides AB and AD have been divided into segments of lengths $1, 2, 3, \ldots, n$. Thus the side of the square has length $n(n+1)/2$ so the area is $[n(n+1)/2]^2$. But the area is also the sum of the areas of the n "gnomons" $G_1, G_2, \ldots, G_n$ shown in the figure. Show that the area of G_i is i^3 and conclude that

$$\sum_{i=1}^{n} i^3 = \left[\frac{n(n+1)}{2}\right]^2$$
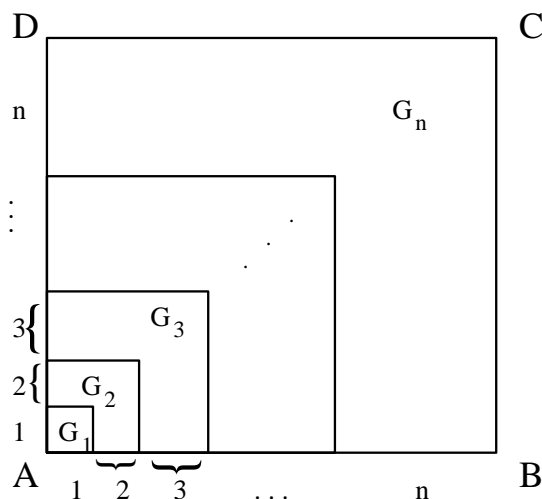
13. (a) Sketch the region in the first quadrant that is bounded by: $f(x) = 16 - x^2$, $y = 0$, $x = 0$.

(b) Approximate this area by calculating:

   i) $\overline{S}_4$

   ii) $\underline{S}_4$

   iii) S_4, where here instead of taking the right-hand or left-hand endpoint of each subinterval, take the midpoint of each subinterval.

(c) Determine $\displaystyle\int_{0}^{4} 16 - x^2 \ dx$.

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14. (a) Sketch the region in the first quadrant that is bounded by the following curves:

$$f(x) = x^3 + 2x^2 + x;$$
$$y = 0;$$
$$x = 1 .$$

(b) Determine the exact area of the region by considering the limit of S_n as $n \to \infty$. Use the right-hand endpoint of each subinterval. Remember that $\sum_{i=1}^{n} i^3 = \left[\frac{n(n+1)}{2}\right]^2$.

15. Express the following limits as definite integrals:

(a) $\displaystyle\lim_{n \to \infty} \sum_{i=1}^{n} \frac{i^4}{n^5}$ [Hint: Consider $f(x) = x^4$.]

(b) $\displaystyle\lim_{n \to \infty} \frac{1}{n} \sum_{i=1}^{n} \frac{1}{1+(\frac{i}{n})^2}$

(c) $\displaystyle\lim_{n \to \infty} \sum_{i=1}^{n} [3(1 + \frac{2i}{n})^5 - 6]\frac{2}{n}$

16. Which of the following expression is NOT equal to $\displaystyle\int_{-a}^{a} x^2 dx$?

(a) $\displaystyle\int_{0}^{2a} (u + a)^2 du$

(b) $\displaystyle 2\int_{0}^{a} x^2 dx$

(c) $\displaystyle -\int_{a}^{-a} x^2 dx$

(d) $\displaystyle\int_{-a}^{a} (x^2 + x)dx$

(e) $\displaystyle\int_{0}^{2a} (u - a)^2 du$

17. Assuming that everything "behaves well" (i.e. that f is continuous and g and h are differentiable functions), find a formula for

$$\frac{d}{dx} \int_{g(x)}^{h(x)} f(t)dt .$$

18. Evaluate the given integrals:

(a) $\displaystyle\int_{-2}^{3} |x^2 - 1| dx$

(b) $\displaystyle\int_{-1}^{2} (x - 2|x|) dx$

(c) $\displaystyle\int_{0}^{2} (x^2 - |x - 1|) dx$

19. *Preamble:* For any (integrable) function f, consider $\displaystyle\int_{a}^{x} f(t)dt$, where a is any constant. By the Fundamental Theorem of Calculus, $\displaystyle\int_{a}^{x} f(t)dt = F(x) - F(a)$, where F is an antiderivative of f. But since $\dfrac{d}{dx}(F(x) - F(a)) = F'(x) = f(x)$ (since $F(a)$ is a constant), then $\dfrac{d}{dx}(\int_{a}^{x} f(t)dt) = f(x)$ and it follows that $\displaystyle\int_{a}^{x} f(t)dt$ **is** an antiderivative of $f(x)$.

(a) Define a function F such that $F'(x) = \dfrac{1}{x}$ and:

    i) $F(2) = 0$

    ii) $F(2) = -3$

(b) Define a function F such that $F'(x) = \sqrt{1 + x^3}$ and:

    i) $F(3) = 0$

    ii) $F(3) = 1$

20. **(a)** Calculate $\int_{0}^{2} 3x^2 \, dx$, and approximate this integral using Simpson's rule for $n = 4$. What is the error?

(b) Given any parabola, $f(x) = ax^2 + bx + c$, can you guess what the error will be using Simpson's rule to approximate $\int_{t}^{s} f(x) \, dx$, given any value of n, the number of subintervals? [Hint: a clue is given on p. 829 of the text. Note: the correct conclusion is *also* true for cubic polynomials $g(x) = ax^3 + bx^2 + cx + d$.]

21. Let T_n denote the approximation of $\int_{a}^{b} f(x) \, dx$ using the trapezoidal rule with n subintervals, $T_{\frac{n}{2}}$ the same but with $\frac{n}{2}$ subintervals (so n must be even), and let S_n denote the approximation of $\int_{a}^{b} f(x) \, dx$ using Simpson's rule with n subintervals.

(a) Show that $S_n = \frac{1}{3}(4T_n - T_{\frac{n}{2}})$.

(b) For the integral $\int_{0}^{2} e^x \, dx$, calculate the integral using the antiderivate, and calculate T_2, T_4 and S_4. Which is most accurate? (At least 3 digits of accuracy is required.)

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22. For a function f such that the derivatives f' and f'' exist for all x in $[a, b]$, the error in approximating $\int_a^b f(x)\, dx$ with the trapezoidal rule with n equal subintervals of length $h = \frac{(b-a)}{n}$ can be approximated by

$$-\frac{h^2}{12}(f'(b) - f'(a))$$

Therefore, an improvised numerical integration rule is given by the corrected trapezoidal rule,

$$\int_a^b f(x)\, dx \approx \frac{h}{2}[f(a) + 2f(a+h) + 2f(a+2h) + \cdots + 2f(a+(n-1)h) + f(b)] - \frac{h^2}{12}[f'(b) - f'(a)]$$

For the integral $\int_0^1 x^{\frac{5}{2}}\, dx$,

    calculate the integral using the antiderivative,

    calculate the approximations given by the trapezoidal rule, corrected trapezoidal rule and Simpson's rule with $n = 4$, and

    compare the errors in the three methods (at least five digits of accuracy are required).

23. The following formulas give the error involved by approximating area by the trapezoidal rule and Simpson's rule.

    If we estimate $\int_a^b f(x)\, dx$ using n subintervals, then the error E is given by:

    For the trapezoidal rule, $E = \frac{(b-a)^3 f''(c)}{12n^2}$ where c is some number between a and b.

    For Simpson's rule, $E = \frac{(b-a)^4 f^{(4)}(c)}{180n^5}$ where c is some number between a and b.

    Determine the values of n for which the absolute value of the error can be guaranteed to be less than $E = 0.01$ using (1) the trapezoidal rule, and (2) Simpson's rule for the following areas.

(a) $\int_1^4 \sqrt{x}\, dx$

(b) $\int_1^3 x^5\, dx$

24. Find the sum of the bounded areas between the x-axis and $f(x) = x^4 + x^3 - 4x^2 - 4x$.

25. The area bounded by the x-axis, the curve $y = f(x)$, and the lines $x = 1$, $x = b$ is equal to

$$\sqrt{b^2 + 1} - \sqrt{2} \text{ for all } b > 1 .$$

Find $f(x)$.

26. (a) Suppose that the quadratic function $f(x) = px^2 + qx + r$ is never negative on $[a, b]$. Show that the area under the graph of f from a to b is

$$A = \frac{1}{3}h[f(a) + 4f(m) + f(b)]$$

where $h = \frac{1}{2}(b - a)$ and $m = \frac{1}{2}(a + b)$. (Hint: By horizontal translation of this region, you can assume that $a = -h$, $m = 0$, and $b = h$.)

(b) As will be seen in HP, Section 17.5, Simpson's Rule states that we can estimate $\int_a^b f(x)dx$ by:

$$\int_a^b f(x)dx \approx \frac{h}{3}\big(f(a) + 4f(a + h) + 2f(a + 2h) + \ldots + 4f(a + (n - 1)h) + f(b)\big)$$

where $h = \frac{(b-a)}{n}$ and n is even. By using part (a), show that if $f(x) = px^2 + qx + r$ is a quadratic function then this "approximation" is in fact precise.

27. Given a curve $y = f(x)$, the length of the curve L from $x = a$ to $x = b$ is given by

$$L = \int_a^b \sqrt{1 + [f'(x)]^2}dx \ .$$

Find the lengths of the following curves over the interval given

(a) $y = (x - 1)^{3/2}$ from $x = 2$ to $x = 10$

(b) $8x^2 = y^3$ from $x = 1$ to $x = 2\sqrt{2}$

(c) $24xy = x^4 + 48$ from $x = 2$ to $x = 3$.

28. Find the area of the region bounded by the two curves by (i) integrating with respect to x and (ii) integrating with respect to y.

(a) $4x + y^2 = 0$, $y = 2x + 4$

(b) $x + 1 = 2(y - 2)^2$, $x + 6y = 7$.

29. Let A and B be the points of intersection of the parabola $y = x^2$ and the line $y = x + 2$, and let C be the point on the parabola where the tangent line is parallel to the graph of $y = x + 2$. Show that the area of the parabolic segment cut off from the parabola by the line (as in the figure below) is four-thirds the area of the triangle ABC.

y=x²

y=x+2

B

A

C
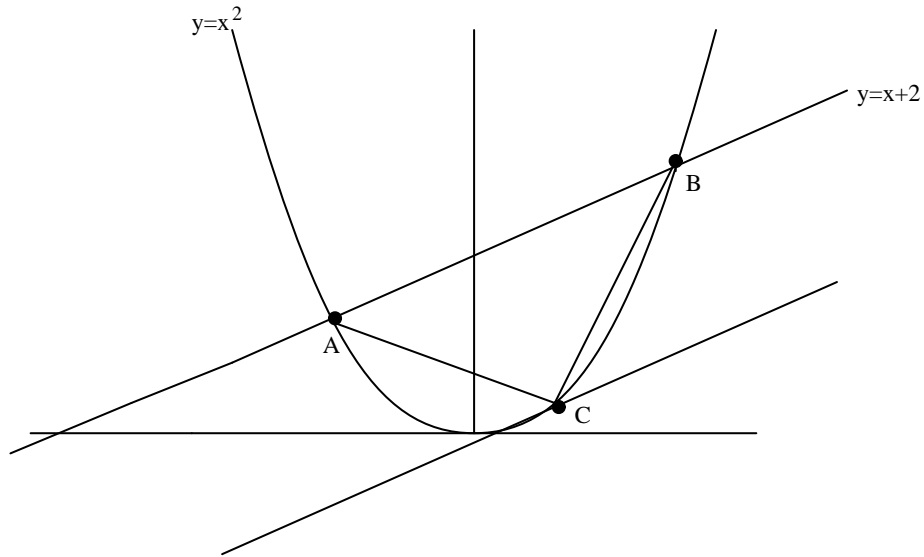
30. Find the area bounded by $y^2 = 4ax$ and $x^2 = 4ay$, where $a > 0$ is a constant.

31. Refer to this graph for the following questions:
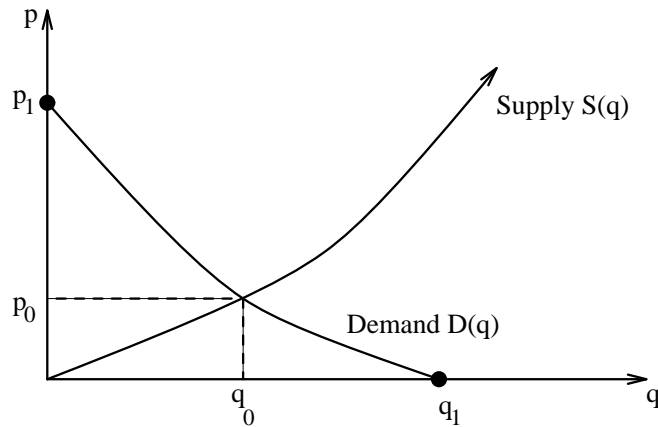
Explain the economic significance of the following:

(a) p_1

(b) q_1

(c) $\displaystyle\int_0^{q_0} S(q)\,dq$

(d) $\displaystyle\int_0^{q_0} D(q)\,dq$

(e) $p_0 q_0$

(f) $\displaystyle\int_0^{q_0} (D(q) - S(q))\,dq$

For (c), (d), (e) & (f), find the appropriate regions that they represent.