

Solutions to Supplementary Questions for HP Chapter 13 and HP Chapter 14.1

1. First, note that $f'(3) = 0$, since f is differentiable everywhere.

- (a) Absolute maximum—since f has no other critical points other than $x = 3$, $f'(x) > 0$ for $x < 3$ (increasing) and $f'(x) < 0$ for $x > 3$ (decreasing), so $f(3)$ is an absolute maximum.
- (b) Absolute minimum—since f has no other critical points except $x = 3$, f must be decreasing for $x < 3$ and increasing for $x > 3$, so $f(3)$ is an absolute minimum.
- (c) Neither—once again, since $x = 3$ is the only critical point, f must be increasing everywhere, indicated by the points given.
- (d) $f'(2) = -1$, and 3 is the only critical point indicates f is decreasing for $x < 3$. $f(3) = 1$, $\lim_{x \rightarrow \infty} f(x) = 3$ and 3 is the only critical point indicates f is increasing for $x > 3$. Therefore $f(3)$ is an absolute minimum.

In all four cases, no conclusions may be reached if there may be other critical points.

2. $f'(x) = abxe^{bx} + ae^{bx} = ae^{bx}(bx + 1)$ Therefore we must have ① $\frac{a}{3}e^{\frac{b}{3}} = 1$ (since $f(\frac{1}{3}) = 1$). Note that $a \neq 0$. Also, ② $ae^{\frac{b}{3}}(\frac{b}{3} + 1) = 0$ since $f'(\frac{1}{3}) = 0$.

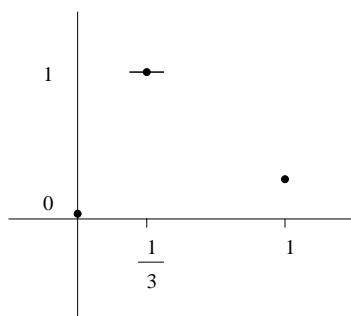
Looking at the second equation, since $e^x > 0$ for all x and since $a \neq 0$ from the first equation, we must have $b = -3$. Looking at the first equation, $\frac{a}{3}e^{-1} = 1$, so $a = 3e$. This shows $f(x) = 3exe^{-3x} = 3xe^{-3x+1}$. The function is differentiable for all x , and above it was shown that $f'(x) = 0$ only when $x = \frac{1}{3}$, so we have

$$f(0) = 0$$

$$f\left(\frac{1}{3}\right) = 1$$

$$f(1) = 3e^{-2} < 1$$

which shows that 1 is an absolute maximum at $x = \frac{1}{3}$. This is because the following three points are on the graph



If $f(\frac{1}{3})$ was not a maximum, we would have another point where the derivative is zero since f is differentiable everywhere. Since there are no other critical points, this is impossible. Therefore, $f(\frac{1}{3})$ is an absolute maximum.

3. The critical points of $f(x)$ occur where the derivative is zero. $f'(x) = 3ax^2 + 2bx + c = 0$. We may solve this equation using the quadratic formula, so the solutions are given by

$$\begin{aligned} x &= \frac{-2b \pm \sqrt{4b^2 - 12ac}}{2a} \\ &= \frac{-b \pm \sqrt{b^2 - 3ac}}{a} \end{aligned}$$

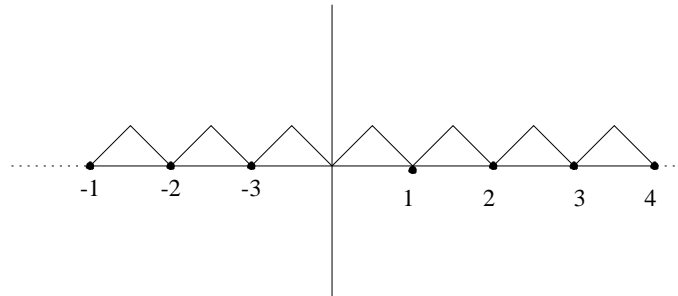
If $b^2 - 3ac < 0$, there are no solutions, so there are no critical points.

If $b^2 - 3ac = 0$, there is one solution giving one critical point.

If $b^2 - 3ac > 0$, there are two solutions giving two critical points.

4. The absolute maximum of $\frac{1}{2}$ occurs at the points $\frac{n}{2}$ where n is an odd integer, and the absolute minimum of 0 occurs at the points n where n is an integer, so these must be critical points.

A sketch of the graph



shows that these are the only critical points.

5. The function may be written as

$$f(x) = \begin{cases} -(x+3) - (x-2); & x < -3 \\ (x+3) - (x-2); & -3 \leq x < 2 \\ (x+3) + (x-2); & x \geq 2 \end{cases}$$

or

$$f(x) = \begin{cases} -2x - 1; & x < -3 \\ 5; & -3 \leq x < 2 \\ 2x + 1; & x \geq 2 \end{cases}$$

$$f'(x) = \begin{cases} -2; & x < -3 \\ 0; & -3 < x < 2 \\ 2; & x > 2 \end{cases}$$

$f'(x)$ is undefined for $x = -3, 2$.

Relative maxima occur for $-3 < x < 2$ and relative minima occur for $-3 \leq x \leq 2$, for which the maximum (and minimum) value is 5. Since there are no other points where $f'(x)$ is undefined or $f'(x) = 0$, 5 is the only relative maximum, and it is also the only relative minimum.

6. For $f(x) = 1 + x - a^x$, $f'(x) = 1 - \ln(a)a^x$, $f'(x) = 0$ when $\ln(a)a^x = 1$.

Case 1 For $0 < a \leq 1$, $\ln(a)a^x \leq 0$ for all x , so there are no critical points. In fact, $f'(x) > 0$ for all x so the function is constantly increasing. In all cases there is the solution $f(0) = 0$ so $1 + x = a^x$ when $x = 0$, and since f is increasing this is the only solution in this case.

Case 2 For $a > 1$, $f'(x) = 0$ when $\ln(a)a^x = 1$,

$$\begin{aligned} a^x &= \frac{1}{\ln(a)} = [\ln(a)]^{-1} \\ x \ln a &= -\ln(\ln(a)) \\ x &= -\frac{\ln(\ln(a))}{\ln(a)} \end{aligned}$$

Let c denote the only critical point $-\frac{\ln(\ln(a))}{\ln(a)}$. When $x = c$, $\ln(a)a^c = 1$, so for $x < c$, $f'(x) = 1 - \ln(a)a^x > 0$ and for $x > c$, $f'(x) = 1 - \ln(a)a^x < 0$. Therefore, the function f has an absolute maximum at $x = c$, and there are no other critical points.

For all a , we know $f(0) = 0$, so the graph of f crosses the x -axis at least once. Since f has an absolute maximum and no other critical points, the graph of f crosses the x -axis at most twice.

Case 2 (a) When $a = e$, $c = \frac{-\ln(\ln e)}{\ln e} = -\ln(1) = 0$ so the solution $x = 0$ is also the point where there is an absolute maximum for f , so the graph touches the x -axis just once, meaning there is only one solution to $e^x = 1 + x$.

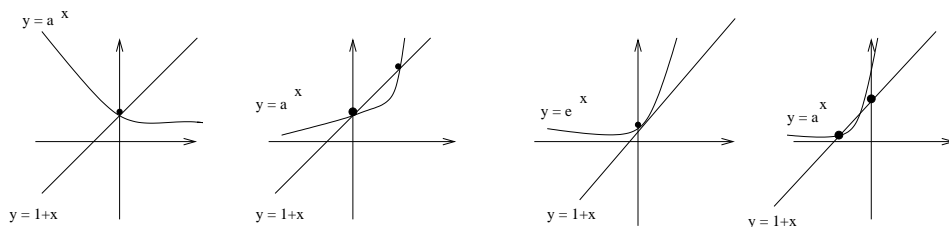
Case 2 (b) When $a \neq e$, $c \neq 0$, so the absolute maximum of f must be above the x -axis since $f(0) = 0$.

Finally, for $1 < a < e$, $f'(0) > 0$ and $\lim_{x \rightarrow \infty} f'(x) = -\infty$, so the function is decreasing at a faster rate as x increases, so it must cross the x -axis for some $x > 0$. For $a > e$, $f'(0) < 0$ and

$$\begin{aligned} \lim_{x \rightarrow -\infty} f(x) &= \lim_{x \rightarrow -\infty} 1 + x - a^x \\ &= \lim_{x \rightarrow -\infty} 1 + x = -\infty \end{aligned}$$

so the function must cross the x -axis for some $x < 0$. Therefore for $a > 1$, $a \neq e$ there are two solutions.

The solutions in the various cases are shown as points of intersection in the following graphs:



Case 1: $0 < a \leq 1$ Case 2(b): $1 < a < e$ Case 2(a): $a = e$ Case 2(b): $a > e$

For $0 < a \leq 1$ or $a = e$ there is one solution to $1 + x = a^x$. For $1 < a < e$ or $a > e$ there are two solutions to $1 + x = a^x$.

7. (a) Intervals on which f is increasing/decreasing—first we differentiate $f(x)$:

$$f'(x) = 1 + \frac{d}{dx}(|x|^{\frac{1}{2}}) = 1 + \frac{1}{2|x|^{\frac{1}{2}}} \left(\frac{d}{dx}|x| \right)$$

But what is $\frac{d}{dx}|x|$?

$$|x| = \begin{cases} x & x \geq 0 \\ -x & x < 0 \end{cases}$$

So

$$\frac{d}{dx}|x| = \begin{cases} 1 & x > 0 \\ -1 & x < 0 \end{cases} \text{ note that } \frac{d}{dx}|x| \text{ is undefined at } x = 0.$$

Hence

$$\frac{d}{dx}|x| = \text{sgn}(x) \quad (x \neq 0), \quad \text{where} \quad \text{sgn}(x) = \begin{cases} 1 & x > 0 \\ 0 & x = 0 \\ -1 & x < 0 \end{cases}$$

$$\text{Hence } f'(x) = 1 + \frac{\text{sgn}(x)}{2\sqrt{|x|}}.$$

Now we search for critical values where either $f'(x)$ is

1) undefined; Note that when $x = 0$, $f'(x)$ is undefined, but otherwise $f'(x)$ is defined; or,

2) $f'(x) = 0$. Solving, we get $f'(x) = 1 + \frac{\text{sgn}(x)}{2\sqrt{|x|}} = 0$.

$\Rightarrow \text{sgn}(x) = -2\sqrt{|x|}$ Note that RHS is negative ($x \neq 0$) so this leaves only one possibility for LHS.

$$\Rightarrow \text{sgn}(x) = -1 = -2\sqrt{|x|}$$

$$\Rightarrow \sqrt{|x|} = \frac{1}{2}$$

$\Rightarrow |x| = \frac{1}{4}$, and remembering that $\text{sgn}(x) = -1$, then
 $\Rightarrow x = -\frac{1}{4}$.

So we get $x = -\frac{1}{4}, 0$ as critical values.

- 1) For $(-\infty, -\frac{1}{4})$, let $x = -4$. Here, $f'(-4) = 1 - \frac{1}{2(2)} = \frac{3}{4}$
- 2) For $(-\frac{1}{4}, 0)$, let $x = -\frac{1}{16}$. Here, $f'(-\frac{1}{16}) = 1 - \frac{1}{2(\frac{1}{4})} = -1$
- 3) For $(0, \infty)$, let $x = 1$. Here, $f'(1) = 1 + \frac{1}{2(1)} = \frac{3}{2}$

So we conclude that $f(x)$ is increasing on $(-\infty, -\frac{1}{4})$ and $(0, \infty)$, and $f(x)$ is decreasing on $(-\frac{1}{4}, 0)$.

(b) Intervals where f is concave down/up. First, we calculate $f''(x)$.

$$\begin{aligned} f'(x) = 1 + \frac{1 + \text{sgn}(x)}{2\sqrt{|x|}} &\Rightarrow f''(x) = \frac{-\text{sgn}(x)}{2(2)} |x|^{-\frac{3}{2}} \left(\frac{d}{dx}|x|\right) \\ &= \frac{-\text{sgn}(x)}{4} |x|^{-\frac{3}{2}} (\text{sgn}(x)) \\ &= \frac{-1}{4|x|^{\frac{3}{2}}} \end{aligned}$$

We search for points where f'' is 0 or not defined.

But $f''(x)$ is nowhere zero, and the only place where f'' is not defined is at $x = 0$. So we need only check below and above $x = 0$.

- 1) For $(-\infty, 0)$, let $x = -1$. Here, $f''(-1) = -\frac{1}{4}$
- 2) For $(0, \infty)$, let $x = 1$. Here, $f''(1) = -\frac{1}{4}$.

Hence f is concave down everywhere, except at $x = 0$.

(c) Relative maxima and minima for f . From (a), there are only two critical points.

- 1) $x = -\frac{1}{4}$. Since f is increasing on $(-\infty, -\frac{1}{4})$ and decreasing on $(-\frac{1}{4}, 0)$, then $x = -\frac{1}{4}$ is a relative maximum (alternatively, f is concave down around $x = -\frac{1}{4}$).
- 2) $x = 0$. Since f is decreasing on $(-\frac{1}{4}, 0)$ and increasing on $(0, \infty)$, then since f is continuous at $x = 0$, $x = 0$ is a relative minimum.

(d) Inflection points of f .

Since f is concave down everywhere (except $x = 0$) there are obviously no inflection points.

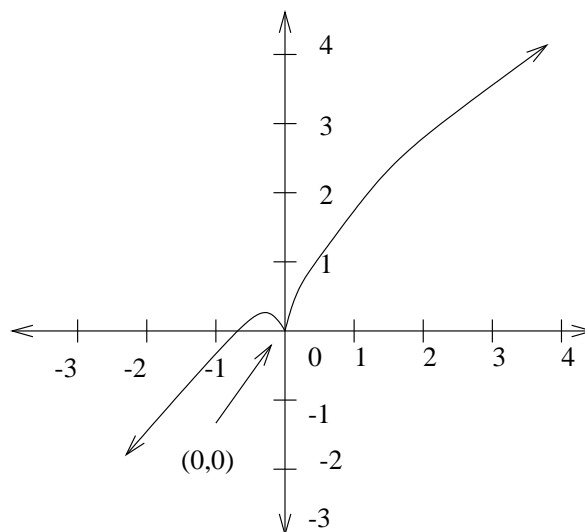
(e) Symmetries of f : none.

(f) Convenient intercepts.

First, at $x = -\frac{1}{4}$, $f(-\frac{1}{4}) = -\frac{1}{4} + \sqrt{|-\frac{1}{4}|} = -\frac{1}{4} + \frac{1}{2} = \frac{1}{4}$. Also, solving $x + \sqrt{|x|} = 0 \Rightarrow x = -\sqrt{|x|}$. (Since RHS is not positive, then $x \leq 0$.)

$$\Rightarrow |x| = x^2 \Rightarrow x = -1, 0$$

Combining all of the above problems to graph $f(x) = x + \sqrt{|x|}$, we get



8. (a) $g'(x) = \frac{-1f'(x)}{(f(x))^2}$ so $g'(x) \leq 0$ when $f'(x) \geq 0$ so $g(x)$ is decreasing in the interval around x_0 .

(b) From (a) we see that $g'(x)$ always has the opposite sign of $f'(x)$, and $g'(x) = 0$ when $f'(x) = 0$. Therefore, by the first derivative test, g has a local minimum at x_1 .

(c)

$$g''(x) = \frac{2(f'(x))^2 f(x) - f''(x)(f(x))^2}{(f(x))^4}$$

Since we don't know the values of $f'(x_2)$ and $f(x_2)$, we cannot conclude whether $g''(x_2)$ is less than, equal to, or greater than, zero, and therefore we cannot obtain any conclusions about the concavity of g at x_2 .

9.

$$\begin{aligned}
f(x) &= \frac{k-x}{x^2+k^2} \\
f'(x) &= \frac{-x^2-k^2-2x(k-x)}{(x^2+k^2)^2} = \frac{x^2-2xk-k^2}{(x^2+k^2)^2} \\
f''(x) &= \frac{(2x-2k)(x^2+k^2)^2-4x(x^2+k^2)(x^2-2xk-k^2)}{(x^2+k^2)^4} \\
&= \frac{2(x^2+k^2)((x-k)(x^2+k^2)-2x(x^2-2xk-k^2))}{(x^2+k^2)^4} \\
&= \frac{2(x^3-k^2x^2+kx^2-k^3-2x^3+4kx^2+2k^2x)}{(x^2+k^2)^3} \\
&= \frac{-2(x^3-3kx^2-3k^2x+k^3)}{(x^2+k^2)^3} \\
&= \frac{-2(x+k)(x^2-4kx+k^2)}{(x^2+k^2)^3} \\
&= \frac{-2(x+k)(x-k(2+\sqrt{3}))(x-k(2-\sqrt{3}))}{(x^2+k^2)^3}
\end{aligned}$$

Since $k \neq 0$, $\frac{-2}{(x^2+k^2)^3} < 0$ is defined for all x and hence $f''(x) = 0$ precisely when $x = -k, k(2+\sqrt{3}), k(2-\sqrt{3})$. Also, it is obvious that these three points are inflection points since $f''(x)$ obviously changes sign as it passes through each of these three points. When $x = -k$,

$$f(-k) = \frac{k - (-k)}{(-k)^2 + k^2} = \frac{1}{k}$$

When $x = k(2+\sqrt{3})$,

$$\begin{aligned}
f(k(2+\sqrt{3})) &= \frac{k - (k(2+\sqrt{3}))}{(k(2+\sqrt{3}))^2 + k^2} \\
&= \frac{-(1+\sqrt{3})}{k(4+4\sqrt{3}+3+1)} = \frac{-(1+\sqrt{3})(2-\sqrt{3})}{4k(2+\sqrt{3})(2-\sqrt{3})} \\
&= \frac{-(2+2\sqrt{3}-\sqrt{3}-3)}{4k} = \frac{1-\sqrt{3}}{4k}
\end{aligned}$$

When $x = k(2-\sqrt{3})$,

$$\begin{aligned}
f(k(2-\sqrt{3})) &= \frac{k - (k(2-\sqrt{3}))}{(k(2-\sqrt{3}))^2 + k^2} \\
&= \frac{-(1-\sqrt{3})}{k(4-4\sqrt{3}+3+1)} = \frac{-(1-\sqrt{3})(2+\sqrt{3})}{4k(2-\sqrt{3})(2+\sqrt{3})} \\
&= \frac{-(2-2\sqrt{3}+\sqrt{3}-3)}{4k} = \frac{1+\sqrt{3}}{4k}
\end{aligned}$$

Using the first two inflection points $(-k, \frac{1}{k})$ and $(k(2+\sqrt{3}), \frac{1-\sqrt{3}}{4k})$, we compute the straight line connecting these points. The slope is:

$$\frac{\frac{1-\sqrt{3}}{4k} - \frac{1}{4k}}{k(2+\sqrt{3}) - (-k)} = \frac{-(3+\sqrt{3})}{4k^2(3+\sqrt{3})} = \frac{-1}{4k^2}$$

so the equation of the straight line is:

$$y - \frac{1}{k} = \frac{-1}{4k^2}(x - (-k)) \Rightarrow y = \frac{-x}{4k^2} + \frac{3}{4k}$$

Plugging in $x = k(2 - \sqrt{3})$, we have

$$\begin{aligned} y &= \frac{-(k(2 - \sqrt{3}))}{4k^2} + \frac{3}{4k} \\ &= \frac{-2 + \sqrt{3}}{4k} + \frac{3}{4k} \\ &= \frac{1 + \sqrt{3}}{4k} \\ &= f(k(2 - \sqrt{3})) \end{aligned}$$

So the third inflection point lies on this line also. Hence, all inflection points of $f(x) = \frac{k-x}{x^2+k^2}$ lie on the same line.

10. $f'(x) = -\frac{2}{5}(x-1)^{-\frac{3}{5}}$

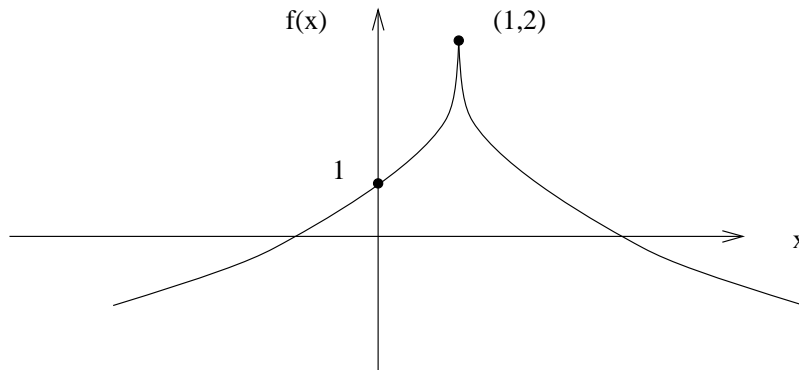
The only critical point is $x = 1$, where $f(1) = 2$. For $x < 1$, $f'(x) > 0$, for $x > 1$, $f'(x) < 0$. Therefore 2 is an absolute maximum at $x = 1$.

$$f''(x) = \frac{6}{25}(x-1)^{-\frac{8}{5}}$$

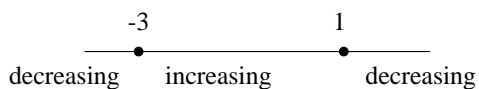
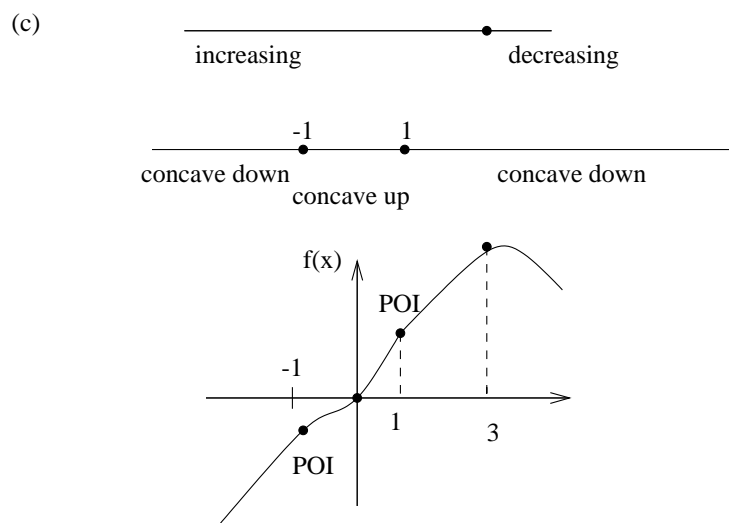
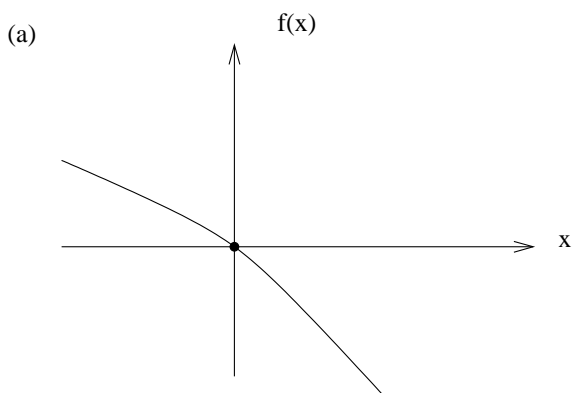
For $x < 1$, $f''(x) > 0$, for $x > 1$, $f''(x) > 0$ so the graph is concave up everywhere.

The y -intercept is $(0, 1)$. There are no asymptotes since the function is continuous and $\lim_{x \rightarrow \infty} f(x) = \lim_{x \rightarrow -\infty} f(x) = -\infty$.

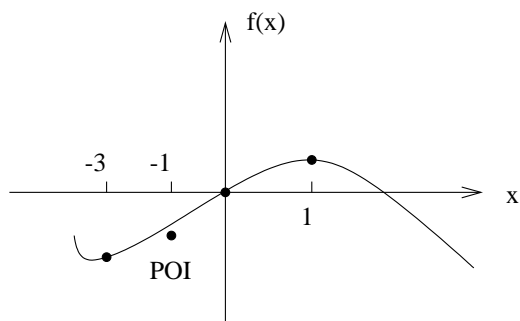
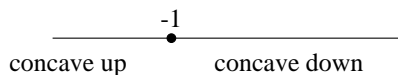
Finally, note that $\lim_{x \rightarrow 1^-} f'(x) = \infty$, $\lim_{x \rightarrow 1^+} f'(x) = -\infty$, so there is a vertical tangent line at $x = 1$. Based on the above information, the graph has the following sketch



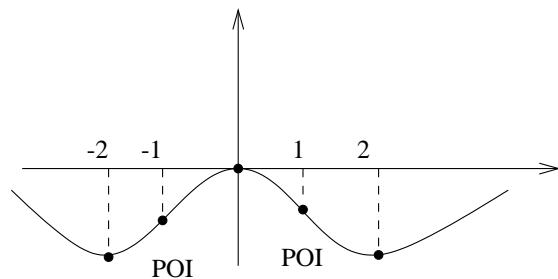
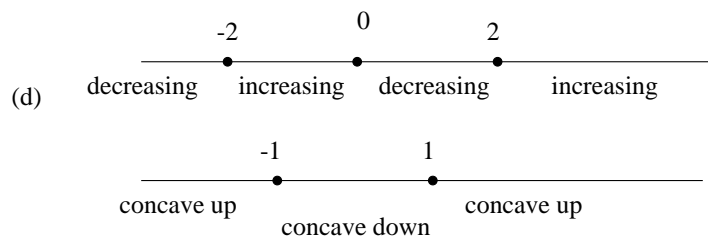
11. (a) Decreasing everywhere, concavity is a negative constant everywhere.



(b)



POI: point of inflection



12. (a) $f'(x) = 3(x^2 - 1)^2 2x = 6x(x^2 - 1)^2$. The critical points are $x = 0, 1, -1$.

$$\begin{aligned} f''(x) &= 6(x^2 - 1)^2 + 6x(2x)(2)(x^2 - 1) \\ &= 6(x^2 - 1)(x^2 - 1 + 4x^2) \\ &= 6(x^2 - 1)(5x^2 - 1) \end{aligned}$$

The second derivative test shows $x = 0$ is a relative minimum.

$$\begin{aligned}f'''(x) &= 6(x^2 - 1)(10x) + 6(5x^2 - 1)(2x) \\&= 6(10x^3 - 10x + 10x^3 - 2x) \\&= 12(10x^3 - 6x)\end{aligned}$$

Now, $f'''(-1) \neq 0$ and $f'''(1) \neq 0$. The above test shows that neither $x = 1$ nor $x = -1$ are relative minima or maxima.

(b)

$$\begin{aligned}f'(x) &= (x - 2)^3(x - 1) + (x - 2)^4 \\&= (x - 2)^3(4x - 4 + x - 2) \\&= (x - 2)^3(5x - 6)\end{aligned}$$

The critical points are $x = 2, \frac{6}{5}$.

$$\begin{aligned}f''(x) &= 3(x - 2)^2(5x - 6) + 5(x - 2)^3 \\&= (x - 2)^2(15x - 18 + 5x - 10) \\&= 4(x - 2)^2(5x - 7)\end{aligned}$$

The second derivative test shows $x = \frac{6}{5}$ is a relative maximum.

$$\begin{aligned}f'''(x) &= 8(x - 2)(5x - 7) + 4(x - 2)^2(5) \\&= 4(x - 2)(10x - 14 + 5x - 10) \\&= 12(x - 2)(5x - 8)\end{aligned}$$

Since $f'''(2) = 0$, we compute another derivative

$$\begin{aligned}f^{(4)}(x) &= 12(5x - 8) + 60(x - 2) \\&= 12(5x - 8 + 5x - 10) \\&= 24(5x - 9)\end{aligned}$$

Since 3 is odd and $f^{(4)}(2) = 24 > 0$, $f(2)$ is a relative minimum.

13. (a) When the number of sets sold (in one week) increases by 100, then the price decreases by \$10. Hence, $p(q) = c - \frac{q}{10}$ for some constant c .

But $p(1000) = 450$, so $450 = c - \frac{1000}{10} \Rightarrow c = 550$. So $p(q) = 550 - \frac{q}{10}$. (Note that by the wording of the problem, we may wish to impose the restriction that $q \geq 1000$.)

(b) Revenue is $r(q) = qp(q) = q(550 - \frac{q}{10}) = 550q - \frac{q^2}{10}$. We find $r'(q) = 550 - \frac{q}{5}$. Letting this be zero gives $550 - \frac{q}{5} = 0 \Rightarrow q = 2,750$. This is a maximum, since $r''(q) = -\frac{1}{5}$ is

negative everywhere. Now, when $q = 2,750$, this is 1,750 T.V. sets above the normal amount of 1000, and hence the rebate that maximizes revenue is $\frac{1,750}{10} = \$175$.

(c) Let $Pr(q)$ be the profit function. Obviously,

$$Pr(q) = r(q) - C(q) = \left[550q - \frac{q^2}{10} \right] - [68,000 + 150q]$$

So $Pr'(q) = 550 - \frac{q}{5} - 150 = 400 - \frac{q}{5}$. Setting this to zero gives $q = 2,000$. This is a maximum since $Pr''(q) = -\frac{1}{5}$ is negative everywhere.

When $q = 2,000$, this is 1,000 T.V. sets above the normal amount of 1,000 and hence the rebate that maximizes profits is $\frac{1,000}{10} = \$100$.

14. (a) There are no vertical asymptotes since the function is continuous.

$$\begin{aligned} \lim_{x \rightarrow \infty} (\sqrt{x^2 + 4} - \sqrt{x^2 - 1}) &= \lim_{x \rightarrow \infty} \frac{x}{x} (\sqrt{x^2 + 4} - \sqrt{x^2 - 1}) \\ &= \lim_{x \rightarrow \infty} x \sqrt{\frac{x^2}{x^2} + \frac{4}{x^2}} - x \sqrt{\frac{x^2}{x^2} - \frac{1}{x^2}} \\ &= \lim_{x \rightarrow \infty} x \sqrt{1 + \frac{4}{x^2}} - x \sqrt{1 - \frac{1}{x^2}} \\ &= \lim_{x \rightarrow \infty} (x - x) \\ &= 0 \end{aligned}$$

Similarly, $\lim_{x \rightarrow -\infty} (\sqrt{x^2 + 4} - \sqrt{x^2 - 1}) = 0$ so there is an horizontal asymptote for ∞ and $-\infty$ at $y = 0$. (Notice that in this case you have to use $\frac{|x|}{|x|}$ instead of $\frac{x}{x}$ in the calculation.)

(b) The denominator is zero when $x = -\frac{e}{a}$ so there is a vertical asymptote at $x = -\frac{e}{a}$,

and

$$\begin{aligned}
\lim_{x \rightarrow -\frac{e}{d}^-} f(x) &= -\infty \\
\lim_{x \rightarrow -\frac{e}{d}^+} f(x) &= +\infty \\
\lim_{x \rightarrow \infty} \frac{\sqrt{ax^2 + bx + c}}{dx + e} &= \lim_{x \rightarrow \infty} \frac{\frac{\sqrt{ax^2 + bx + c}}{x}}{\frac{dx + e}{x}} \\
&= \lim_{x \rightarrow \infty} \frac{\sqrt{\frac{ax^2}{x^2} + \frac{bx}{x^2} + \frac{c}{x^2}}}{\frac{dx}{x} + \frac{e}{x}} \\
&= \lim_{x \rightarrow \infty} \frac{\sqrt{a + \frac{b}{x} + \frac{c}{x^2}}}{d + \frac{e}{x}} \\
&= \frac{\sqrt{a}}{d} \\
\lim_{x \rightarrow -\infty} \frac{\sqrt{ax^2 + bx + c}}{dx + e} &= \lim_{x \rightarrow -\infty} \frac{\frac{\sqrt{ax^2 + bx + c}}{x}}{\frac{dx + e}{x}} \\
&= \lim_{x \rightarrow -\infty} \frac{-\sqrt{\frac{ax^2}{x^2} + \frac{bx}{x^2} + \frac{c}{x^2}}}{\frac{dx}{x} + \frac{e}{x}} \\
&= \lim_{x \rightarrow -\infty} \frac{-\sqrt{a + \frac{b}{x} + \frac{c}{x^2}}}{d + \frac{e}{x}} \\
&= -\frac{\sqrt{a}}{d}
\end{aligned}$$

Therefore at $+\infty$ there is a horizontal asymptote at $y = \frac{\sqrt{a}}{d}$ and at $-\infty$ there is a horizontal asymptote at $y = -\frac{\sqrt{a}}{d}$

15. (a) For horizontal asymptotes we check

$$\lim_{x \rightarrow \infty} \ln \left(\frac{1}{|x|} \right) = \lim_{x \rightarrow -\infty} \ln \left(\frac{1}{|x|} \right) = -\infty$$

so there are no horizontal asymptotes. There is a vertical asymptote at $x = 0$ since $\lim_{x \rightarrow 0} \ln \left(\frac{1}{|x|} \right) = \infty$.

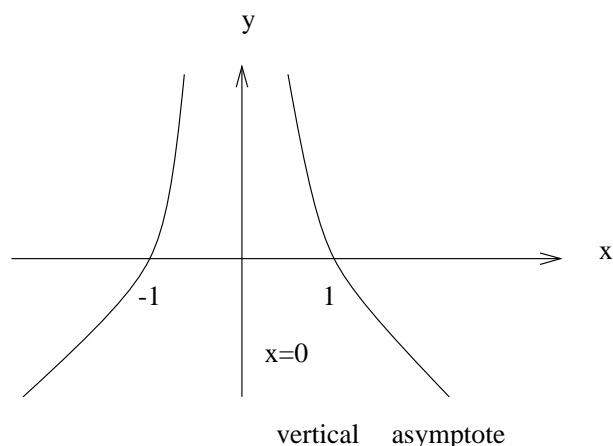
For positive values of x , we see

$$\frac{d}{dx} \ln \left(\frac{1}{x} \right) = x \left(-\frac{1}{x^2} \right) = -\frac{1}{x} < 0$$

So for positive values of x the function is decreasing. The graph is symmetrical about the y -axis since

$$\ln \left(\frac{1}{|x|} \right) = \ln \left(\frac{1}{|-x|} \right)$$

Based on the above information, we can sketch the graph



(b) $y = e^{\frac{1}{x}}$. For horizontal asymptotes we check

$$\lim_{x \rightarrow \infty} e^{\frac{1}{x}} = e^0 = 1$$

$$\lim_{x \rightarrow -\infty} e^{\frac{1}{x}} = e^0 = 1$$

so there is a horizontal asymptote for ∞ and $-\infty$ at $y = 1$. There is a vertical asymptote at $x = 0$ since

$$\lim_{x \rightarrow 0^+} e^{\frac{1}{x}} = \infty$$

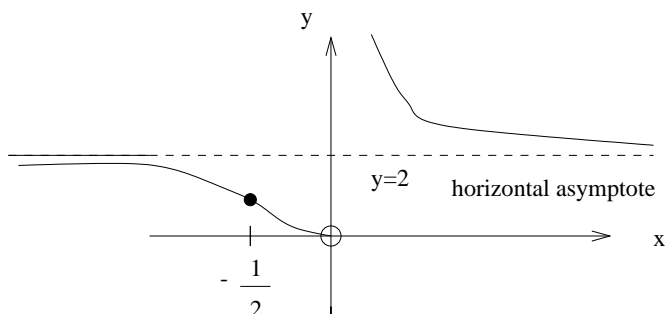
but note that

$$\lim_{x \rightarrow 0^-} e^{\frac{1}{x}} = 0$$

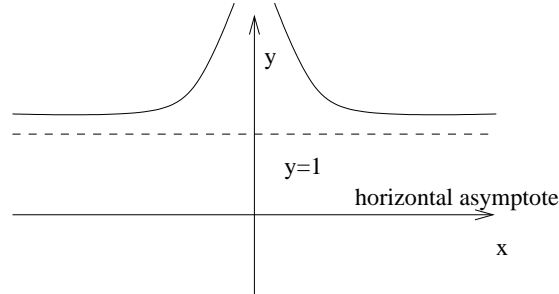
The graph is always decreasing since

$$\frac{d}{dx} e^{\frac{1}{x}} = e^{\frac{1}{x}} \left(-\frac{1}{x^2} \right) = -\frac{e^{\frac{1}{x}}}{x^2} < 0$$

Since $\frac{d^2}{dx^2} (e^{\frac{1}{x}}) = (e^{\frac{1}{x}}) = \frac{1}{x^2} \left(\frac{e^{\frac{1}{x}}}{x^2} \right) + \frac{2e^{\frac{1}{x}}}{x^3} = \frac{e^{\frac{1}{x}}}{x^3} \left(\frac{1}{x} + 2 \right)$ we can see that there is a point of inflection at $x = -\frac{1}{2}$ where the second derivative changes from negative to positive. The graph is



- (c) $y = e^{\frac{1}{|x|}}$. The graph will be the same as above for positive values of x , and will be symmetric about the y -axis, so a sketch of the graph is



16.

$$\begin{aligned} f(t) &= k(e^{-at} - e^{-bt}) \\ f'(t) &= k(be^{-bt} - ae^{-at}) \\ f''(t) &= k(a^2e^{-at} - b^2e^{-bt}) \end{aligned}$$

$f'(t) = 0$ when

$$\begin{aligned} be^{-bt} &= ae^{-at} \Rightarrow \frac{b}{a} = e^{(b-a)t} \\ &\Rightarrow \ln\left(\frac{b}{a}\right) = (b-a)t \\ &\Rightarrow t = \frac{\ln\left(\frac{b}{a}\right)}{b-a} \end{aligned}$$

Note that since $b > a$, $b - a > 0$, $\ln\left(\frac{b}{a}\right) > 0$, so $\frac{\ln\left(\frac{b}{a}\right)}{b-a} > 0$

Similarly, $f''(t) = 0$ when $t = \frac{\ln\left(\frac{a^2}{b^2}\right)}{(a-b)} = \frac{\ln\left(\frac{b}{a}\right)^{-2}}{-(b-a)} = \frac{2\ln\left(\frac{b}{a}\right)}{b-a}$.

Since $b > a$, $f''(t) > 0$ when $t > \frac{2\ln\left(\frac{b}{a}\right)}{b-a}$ and $f''(t) < 0$ when $t < \frac{2\ln\left(\frac{b}{a}\right)}{b-a}$ so $t = \frac{2\ln\left(\frac{b}{a}\right)}{b-a}$ is an inflection point where the concavity changes from negative to positive.

We now use the second derivative test on the only critical point $t = \frac{\ln(\frac{b}{a})}{b-a}$.

$$\begin{aligned}
 f''\left(\frac{\ln(\frac{b}{a})}{b-a}\right) &= k\left(a^2 e^{-\frac{a}{b-a} \ln(\frac{b}{a})} - b^2 e^{-\frac{b}{b-a} \ln(\frac{b}{a})}\right) \\
 &= k\left(a^2 \left(\frac{b}{a}\right)^{-\frac{a}{b-a}} - b^2 \left(\frac{b}{a}\right)^{-\frac{b}{b-a}}\right) \\
 &= k\left(\frac{b}{a}\right)^{-\frac{a}{b-a}} \left[a^2 - b^2 \left(\frac{b}{a}\right)^{\frac{a-b}{b-a}} \right] \\
 &= k\left(\frac{b}{a}\right)^{-\frac{a}{b-a}} \left[a^2 - b^2 \left(\frac{b}{a}\right)^{-1} \right] \\
 &= k\left(\frac{b}{a}\right)^{-\frac{a}{b-a}} [a^2 - ba] \\
 &= ak\left(\frac{b}{a}\right)^{-\frac{a}{b-a}} (a - b) < 0
 \end{aligned}$$

since $a, b, k > 0$ and $b > a$. Therefore $t = \frac{\ln(\frac{b}{a})}{b-a}$ is a relative maximum. Also, since it is the only critical point of the function, it is an absolute maximum and there is no other relative maxima or minima.

Finally, we check asymptotes. The function is continuous, so there are no vertical asymptotes.

$$\begin{aligned}
 \lim_{t \rightarrow \infty} f(t) &= \lim_{t \rightarrow \infty} k(e^{-at} - e^{-bt}) = \lim_{t \rightarrow \infty} ke^{-at}(1 - e^{(a-b)t}) \\
 &= \lim_{t \rightarrow \infty} ke^{-at}(1) \quad \text{since } b > a \\
 &= 0 \quad \text{since } a > 0 \\
 \lim_{t \rightarrow -\infty} f(t) &= \lim_{t \rightarrow -\infty} k(e^{-at} - e^{-bt}) = \lim_{t \rightarrow -\infty} ke^{-bt}(e^{(b-a)t} - 1) \\
 &= \lim_{t \rightarrow -\infty} ke^{-bt}(-1) \quad \text{since } b > a \\
 &= -\infty \quad \text{since } b > 0
 \end{aligned}$$

We check a few points to make the sketch easier.

$$\begin{aligned}
 f(0) &= k(e^0 - e^0) = 0 \\
 f\left(\frac{\ln(\frac{b}{a})}{b-a}\right) &= k\left(e^{-\frac{a}{b-a} \ln(\frac{b}{a})} - e^{-\frac{b}{b-a} \ln(\frac{b}{a})}\right) \\
 &= k\left[\left(\frac{b}{a}\right)^{-\frac{a}{b-a}} - \left(\frac{b}{a}\right)^{-\frac{b}{b-a}}\right] \\
 &= k\left(\frac{b}{a}\right)^{-\frac{a}{b-a}} \left[1 - \left(\frac{b}{a}\right)^{\frac{a-b}{b-a}}\right] \\
 &= k\left(\frac{b}{a}\right)^{-\frac{a}{b-a}} \left[1 - \frac{a}{b}\right] > 0 \quad \text{since } b > a
 \end{aligned}$$

Similarly,

$$f\left(\frac{2 \ln\left(\frac{b}{a}\right)}{b-a}\right) = k\left(\frac{b}{a}\right)^{\frac{-2a}{b-a}} \left[1 - \left(\frac{a}{b}\right)^2\right] > 0 \quad \text{since } b > a.$$

Summarizing, we have the following information:

no vertical asymptotes

a horizontal asymptote at $y = 0$ for $+\infty$, $\lim_{t \rightarrow -\infty} f(t) = -\infty$

an absolute maximum at $t = \frac{\ln\left(\frac{b}{a}\right)}{b-a} > 0$,

$$f\left(\frac{\ln\left(\frac{b}{a}\right)}{b-a}\right) = k\left(\frac{b}{a}\right)^{\frac{-a}{b-a}} \left[1 - \frac{a}{b}\right] > 0$$

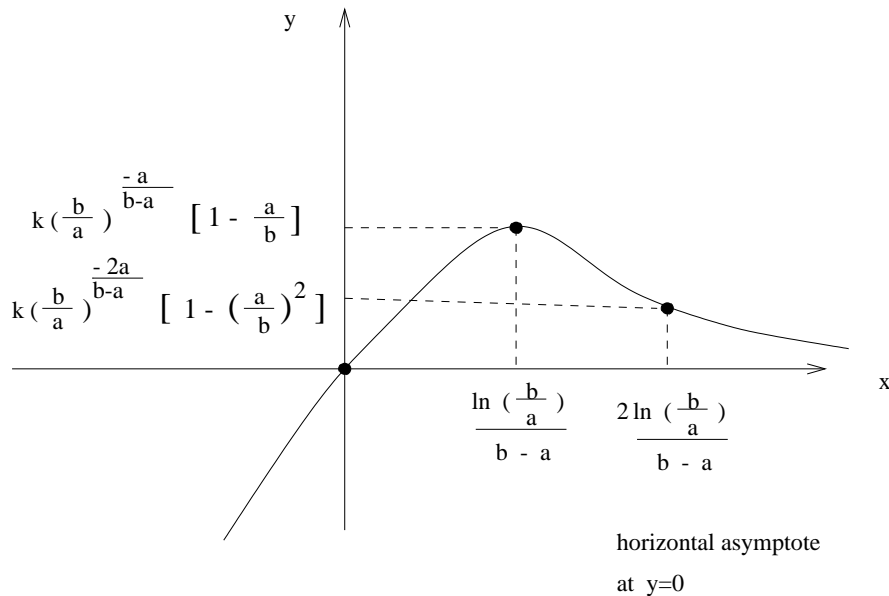
and there are no other relative extrema.

A point of inflection at $t = \frac{2 \ln\left(\frac{b}{a}\right)}{b-a} > 0$. The graph is concave down before this point and concave up afterwards.

$$f\left(\frac{2 \ln\left(\frac{b}{a}\right)}{b-a}\right) = k\left(\frac{b}{a}\right)^{\frac{-2a}{b-a}} \left[1 - \left(\frac{a}{b}\right)^2\right] > 0$$

$f(0) = 0$.

Based on the above information, we know the graph is the following:



17. The present value now is given by

$$P(t) = (\overbrace{10000e^{\sqrt{t}}})^{\text{market value}} (e^{-0.10t})$$

where t is the year when the real estate is sold.

$$\begin{aligned}
 P(t) &= 10000e^{\sqrt{t}-0.10t} \\
 P'(t) &= 10000e^{\sqrt{t}-0.10t} \left(\frac{1}{2\sqrt{t}} - 0.1 \right) \\
 P'(t) = 0 & \quad \text{when} \quad \frac{1}{2\sqrt{t}} = 0.1 \Rightarrow \sqrt{t} = \frac{1}{0.2} = 5 \Rightarrow t = 25
 \end{aligned}$$

Since $P'(t)$ is positive for $t < 25$ and negative for $t > 25$, so $t = 25$ is an absolute maximum. Therefore, the real estate should be sold 25 years from now.

18. The total yearly expense associated with the machine if it is kept for t years is $C(t) = \frac{p}{t} + \frac{nr}{t}$ where $\frac{p}{t}$ is the average replacement cost. $\frac{nr}{t}$ is the average maintenance cost. Since $n(t) = \frac{t^\alpha}{\beta}$, we have

$$\begin{aligned}
 C(t) &= \frac{p}{t} + \frac{rt^{\alpha-1}}{\beta} \\
 C'(t) &= \frac{-p}{t^2} + \frac{r(\alpha-1)t^{\alpha-2}}{\beta}
 \end{aligned}$$

Solving for when $C'(t) = 0$, we have

$$\begin{aligned}
 0 &= \frac{-p}{t^2} + \frac{r(\alpha-1)t^{\alpha-2}}{\beta} \\
 0 &= -p + \frac{r(\alpha-1)t^\alpha}{\beta} \\
 t^\alpha &= \frac{p\beta}{r(\alpha-1)} \\
 t &= \sqrt[\alpha]{\frac{p\beta}{r(\alpha-1)}}
 \end{aligned}$$

Taking the second derivative, we have

$$C''(t) = \frac{2p}{t^3} + \frac{r(\alpha-1)(\alpha-2)t^{\alpha-3}}{\beta}$$

We know $p > 0$, $t > 0$, $r > 0$, $\alpha \geq 2$, $\beta > 0$, so we may conclude $C''(t) > 0$ for all t . Since this shows that the graph of $C(t)$ is concave up for all t , we may conclude that the critical value $t = \sqrt[\alpha]{\frac{p\beta}{r(\alpha-1)}}$ is an absolute minimum, so the optimal time to replace the machine is in $\sqrt[\alpha]{\frac{p\beta}{r(\alpha-1)}}$ years.

19. The revenue as a function of x is given by

$$\begin{aligned}R(x) &= (10 + x)(10000)(0.92)^x + (9 + x)(20000)(0.92)^x \\ &\quad + (8 + x)(30000)(0.92)^x + (0.33)(60000)(0.92)^x \\ &= (0.92)^x(100000 + 10000x + 180000 + 20000x + 240000 + 30000x \\ &\quad + 19800) \\ &= (0.92)^x(60000x + 539800) \\ &= (0.92)^x(100)(600x + 5398)\end{aligned}$$

Taking the derivative, we have

$$\begin{aligned}R'(x) &= 100[\ln(0.92)(0.92)^x(600x + 5398) + 600(0.92)^x] \\ &= 100(0.92)^x(\ln(0.92)600x + \ln(0.92)5398 + 600)\end{aligned}$$

$R'(x) = 0$ when $\ln(0.92)600x + \ln(0.92)5398 + 600 = 0$, so

$$x = -\frac{\ln(0.92)5398 + 600}{\ln(0.92)600} = 3.$$

Since $\ln(0.92) < 0$, we see that $R'(x) > 0$ for $x < 3$ and $R'(x) < 0$ for $x > 3$. Therefore $x = 3$ is an absolute maximum, so a \$3 increase in ticket prices will maximize revenue.

20. Revenue is given by $R(p) = pq$. Since $q = a - bp$, $p = \frac{a-q}{b}$. Revenue as a function of output is given by

$$\begin{aligned}R(q) &= \left(\frac{a-q}{b}\right)q \\ &= \frac{1}{b}(aq - q^2)\end{aligned}$$

Profit is given by

$$\begin{aligned}P(q) &= R(q) - C(q) \\ &= \frac{1}{b}(aq - q^2) - kq^2 \\ &= \frac{a}{b}q + \left(-\frac{1}{b} - k\right)q^2 \\ P'(q) &= \frac{a}{b} + 2\left(-\frac{1}{b} - k\right)q\end{aligned}$$

$P'(q) = 0$ when $-2\left(-\frac{1}{b} - k\right)q = \frac{a}{b} \Rightarrow q = \frac{a}{2b(k+\frac{1}{b})} = \frac{a}{2(kb+1)}$. Therefore, the only critical point is $q = \frac{a}{2(kb+1)}$.

$$P''(q) = 2\left(-\frac{1}{b} - k\right)$$

Since $k > -\frac{1}{b}$, $P''(q) < 0$ for all q , indicating that $q = \frac{a}{2(kb+1)}$ is an absolute maximum, and is therefore the profit maximizing output.

The price for the output is given by

$$\begin{aligned} p &= \frac{a - q}{b} = \frac{a - \frac{a}{2(kb+1)}}{b} \\ &= \frac{a\left(\frac{2(kb+1)-1}{2(kb+1)}\right)}{b} \\ &= \left(\frac{a}{b}\right) \left(\frac{2(kb+1)-1}{2(kb+1)}\right) \\ &= \left(\frac{a}{b}\right) \left(\frac{2kb+1}{2kb+2}\right) \end{aligned}$$

21. Let

$$\begin{aligned} f(x) &= \sqrt{x} + \sqrt[3]{x} + \sqrt[4]{x} + \sqrt[5]{x} \\ &= x^{\frac{1}{2}} + x^{\frac{1}{3}} + x^{\frac{1}{4}} + x^{\frac{1}{5}} \\ f'(x) &= \frac{1}{2}x^{-\frac{1}{2}} + \frac{1}{3}x^{-\frac{2}{3}} + \frac{1}{4}x^{-\frac{3}{4}} + \frac{1}{5}x^{-\frac{4}{5}} \end{aligned}$$

Let $x = 1$, $dx = 0.02$. By the differential formula,

$$\begin{aligned} f(x + dx) &\approx f(x) + dy = f(x) + f'(x)dx \\ f(1.02) &\approx f(1) + f'(1)(0.02) \\ &= 4 + \left(\frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5}\right)(0.02) \\ &= 4 + \left(\frac{30 + 20 + 15 + 12}{60}\right) \left(\frac{1}{50}\right) \\ &= 4 + \frac{77}{60} \cdot \frac{1}{50} \\ &= 4 + \frac{77}{3000} \\ &= \frac{12000 + 77}{3000} \\ &= \frac{12077}{3000} \\ &\approx 4.026 \end{aligned}$$

22. (a) Let $f(x) = e^x$. From differentials, we have

$$\begin{aligned} f(x + dx) &\approx f(x) + f'(x)dx \\ e^{x+dx} &\approx e^x + e^x dx \end{aligned}$$

Letting $x = 0$ and $dx = kh$, we have

$$e^{kh} \approx e^0 + e^0 kh$$

$$e^{kh} \approx 1 + kh$$

We now apply this to the limit. For small values of kh the approximation becomes more accurate, so

$$\lim_{h \rightarrow 0} \frac{e^{kh} - 1}{h} = \lim_{h \rightarrow 0} \frac{1 + kh - 1}{h} = \lim_{h \rightarrow 0} \frac{kh}{h} = k$$

(b) The limit is $f'(0)$ where $f(x) = e^{kx}$, so $f'(x) = ke^{kx}$ and $f'(0) = ke^{k \cdot 0} = ke^0 = k$ as before.

23. (a) $Q(998) = Q(1000 + (-2)) \approx Q(1000) + (-2)Q'(1000)$, $Q'(L) = \frac{36}{L^{\frac{4}{3}}}$. This gives

$$Q(998) \approx 54(1000)^{\frac{2}{3}} - \frac{2(36)}{(1000)^{\frac{1}{3}}}$$

$$= 54(100) - \frac{72}{10}$$

$$= 5392.8$$

(b) Note that $1331 = 11^3$.

ΔL	$Q(1331 + \Delta L)$	$Q(1331) + Q'(1331)\Delta L$
1	6537.27	6537.27
10	6566.69	6566.73
20	6599.29	6599.45
30	6631.82	6632.18
40	6664.26	6664.91
49	6693.40	6694.36
50	6696.63	6697.64

We see ΔL must be at least 50 for a difference of one unit.

24. The formula for compound interest is $B = P(1+r)^t$ where $r = \frac{i}{100}$, P is the principal, and B is the balance after t years. We wish to solve $2P = P(1+r)^t$ or $2 = (1+r)^t$. Solving for t we have

$$\ln 2 = t \ln(1+r)$$

$$t = \frac{\ln 2}{\ln(1+r)}$$

The difference formula $f(1 + dx) \approx f(1) + dy$ for $f(x) = \ln(x)$ gives

$$f(1 + dx) \approx f(1) + f'(x)dx$$

$$\ln(1 + dx) \approx \ln(1) + \frac{1}{1}dx$$

$$\ln(1 + dx) \approx dx$$

This is accurate for small values of dx , so if r is small, we have $\ln(1 + r) \approx r$.

Thus the solution t is

$$t = \frac{\ln(2)}{\ln(1 + r)} \approx \frac{\ln(2)}{r} \approx \frac{0.693}{r} = \frac{69.3}{i} \approx \frac{70}{i}$$