Supplementary Questions for HP Chapter 13 and HP Chapter 14.1

1. Assume the function f is differentiable everywhere and has just one critical point, at $x = 3$. In (a) through (d), additional conditions are given. In each case, decide whether there is an absolute maximum, absolute minimum, or neither at $x = 3$.

(a) $f'(1) = 3$ and $f'(5) = -1$

(b) $\lim_{x\to\infty} f(x) = \infty$ and $\lim_{x\to-\infty} f(x) = \infty$

(c) $f(1) = 1$, $f(2) = 2$, $f(4) = 4$, $f(5) = 5$

(d) $f'(2) = -1$, $f(3) = 1$, $\lim_{x\to\infty} f(x) = 3$.

2. Choose constants a, b for the function $f(x) = axe^{bx}$ so that $f(\frac{1}{3}) = 1$ and the function has a relative maximum at $x = \frac{1}{3}$.

3. Explain why the cubic polynomial function $f(x) = ax^3 + bx^2 + cx + d$ $(a \neq 0)$ can have either two, one, or no critical points on the real line.

4. Define $f(x)$ to be the distance from x to the nearest integer. What are the critical points of f?

5. Find the relative maximum and minimum values of the function $f(x) = |x+3|+|x-2|$.

6. How many solutions are there to the equation $a^x = 1 + x$ $(a > 0)$? Solve this problem by considering maxima/minima of the function $f(x) = 1 + x - a^x$. The answer will vary depending on the constant a.

7. For the function $f(x) = x + \sqrt{|x|}$, determine:

(a) the intervals on which the function is increasing and decreasing,

(b) where f is concave down and concave up,

(c) relative maxima and minima for f,

(d) inflection points of f,

(e) any symmetries of f,

(f) those intercepts for f that can be obtained conveniently.
(g) Lastly, draw a graph of $f(x)$.

8. Let f be a twice differentiable function (i.e. f has two derivatives everywhere) such that $f(x) \neq 0$ for all x. Let $g = \frac{1}{f}$.

(a) If f is increasing in an interval around x_0, what about g?

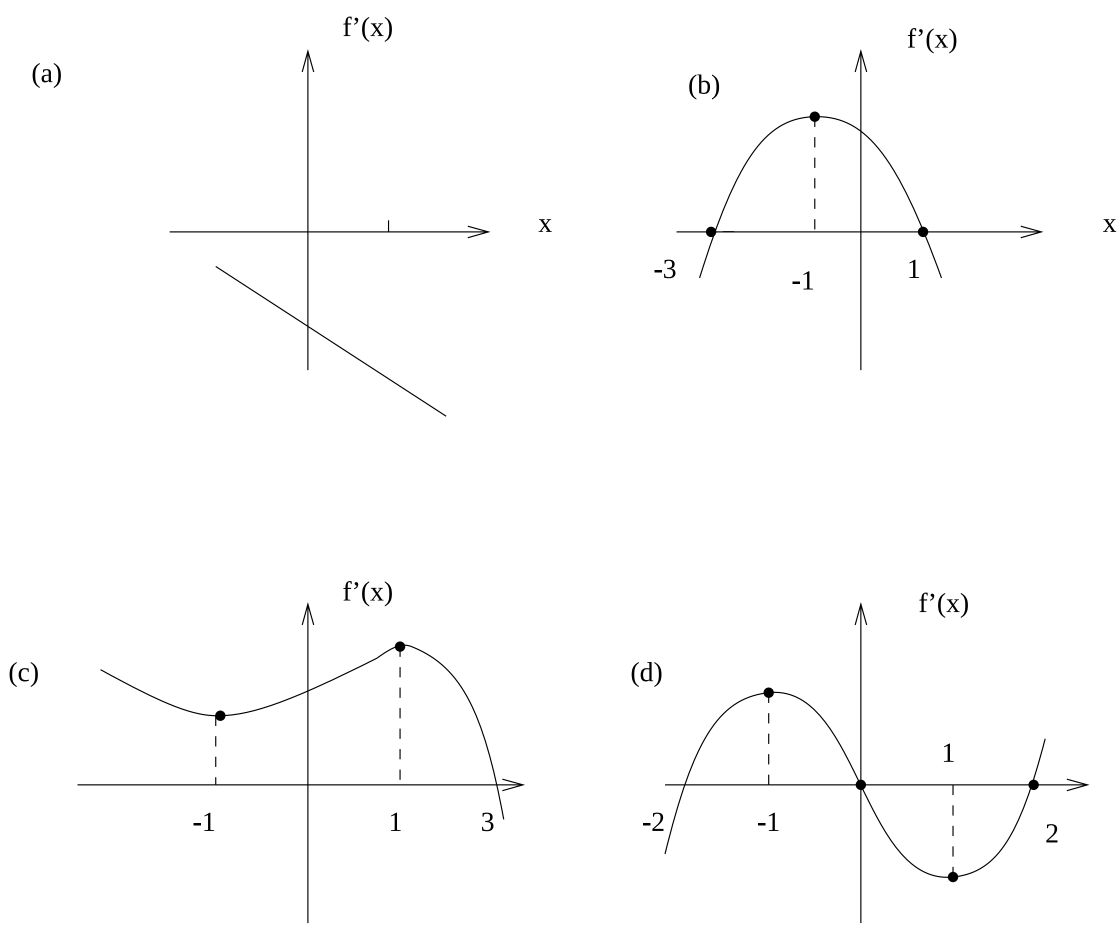
(b) If f has a local maximum at x_1, what about g?

(c) If f is concave down at x_2, what about g?

9. Show that the points of inflection of $f(x) = \frac{k-x}{x^2+k^2}$ (k is a constant, $k \neq 0$) all lie on a straight line.

10. Sketch the graph of the function $f(x) = 2 - (x-1)^{\frac{2}{5}}$.

11. In each of the four graphs below, the graph of the first derivative f' of a function is sketched. In each case, determine whether the function is increasing, decreasing, concave up, and concave down. Put this information on a number line and sketch the graph of the function f, assuming that $f(0) = 0$.



12. The second derivative test fails to work at a point x_0 of a function $f(x)$ if $f'(x_0) = f''(x_0) = 0$. The following can be used in such cases. Suppose f has derivatives of all orders at x_0, the first n derivatives are zero at x_0, but the $(n+1)^{\text{th}}$ derivative is not zero, i.e., $0 = f'(x_0) = f''(x_0) = \ldots = f^{(n)}(x_0)$, $f^{(n+1)} \neq 0$. Then,

(i) if n is even, $f(x_0)$ is neither a maximum or minimum,

(ii) if n is odd, $f^{(n+1)}(x_0) > 0$, $f(x_0)$ is a relative minimum,

(iii) if n is odd, $f^{(n+1)}(x_0) < 0$, $f(x_0)$ is a relative maximum.

Use this test to determine whether critical points of the following functions are relative maxima, minima, or neither.

(a) $f(x) = (x^2 - 1)^3$

(b) $f(x) = (x-2)^4(x-1)$

13. A manufacturer has been selling 1000 television sets a week, at \$450 each. A market survey indicates that for each \$10 rebate offered to the buyer, the number of sets sold will increase by 100 per week.

(a) Find the demand function.

(b) How large a rebate should the company offer the buyer in order to maximize its revenue?

(c) If its weekly cost function is $C(q) = 68,000 + 150q$, how should it set the size of the rebate in order to maximize profits?

14. Determine whether the following functions have horizontal and/or vertical asymptotes.

(a) $f(x) = \sqrt{x^2+4} - \sqrt{x^2-1}$ (Hint: multiply by $\frac{x}{x}$.)

(b) $f(x) = \frac{\sqrt{ax^2+bx+c}}{dx+e}$ where $a > 0$, $d \neq 0$, b, c, e are constants, and $ax^2+bx+c > 0$ for all x, (same hint as above).

15. Sketch graphs of the following curves indicating vertical and horizontal asymptotes.

(a) $y = \ln\left(\frac{1}{|x|}\right)$

(b) $y = e^{\frac{1}{x}}$

(c) $y = e^{\frac{1}{|x|}}$

16. When a drug is injected into the blood at time $t = 0$, it is sometimes assumed that the concentration of the drug thereafter is given by the function $f(t) = k(e^{-at} - e^{-bt})$ where k, a, b are positive constants, $b > a$. Sketch a graph of this function.

17. Suppose you own real estate, for which the market value t years from now is given by $V(t) = 10000e^{\sqrt{t}}$. Suppose the interest rate is 10% compounded continuously. When is the optimal time to sell (so that current present value is maximized)?

18. A rental company buys a new machine for p dollars, which it then rents to customers. If the company keeps the machine for t years (before replacing it), the average replacement cost per year for the t years is $\frac{p}{t}$. During these t years, the company must make repairs on the machine, the number of repairs n depending on t as given by $n(t) = \frac{t^\alpha}{\beta}$ where $\alpha \geq 2$ and $\beta > 0$ are constants. If r is the average cost per repair, then the average maintenance cost per year over the life of the machine is $\frac{nr}{t}$. Find the optimum time to replace the machine, minimizing the average yearly expense of the machine.

19. A football team presently sells tickets at prices of \$8, \$9 and \$10 per seat, depending on the position of the seat. At these prices they average sales of 10000 at \$10, 20000 at \$9, and 30000 at \$8. The team also receives an average of 33 cents per person in concession sales. The team wishes to raise the amount of each ticket by the same amount of x dollars, but feels that will result in average ticket sales of $(0.92)^x(10000)$ at $\$(10 + x)$, $(0.92)^x(20000)$ at $\$(9 + x)$ and $(0.92)^x(30000)$ at $\$(8 + x)$. Find x so that revenue is maximized.

20. For the demand function $q = a - bp$ (where q is output, p is price, $a, b > 0$ are constants), and the cost function $C(q) = kq^2$, (where C is cost, $k > 0$ is a constant), calculate the formula for the optimal (profit maximizing) output and its price, where $k > \frac{-1}{b}$.

21. Use differentials to find an approximation to the number $\sqrt[2]{1.02} + \sqrt[3]{1.02} + \sqrt[4]{1.02} + \sqrt[5]{1.02}$.

22. **(a)** Using differentials, evaluate $\lim_{h\to 0} \frac{e^{kh}-1}{h}$ by letting $dx = kh$.

(b) Do you recognize the limit as something familiar?

23. Consider the function $Q(L) = 54L^{\frac{2}{3}}$ which relates units of production Q for a company to units of labour, L.

(a) Using differentials, estimate Q when L is 998.

(b) If we write $Q(1331 + \Delta L) \approx Q(1331) + Q'(1331)\Delta L$ (the differential formula), except that we consider positive integer values of ΔL only, how big must ΔL be in order for $Q(1331 + \Delta L)$ and its approximation $Q(1331) + Q'(1331)\Delta L$ to differ by more than one unit? (Plug increasing values of ΔL into these two expressions.)

24. The 'Rule of 70' says that the time it takes for money to double in an account bearing $i\%$ annual interest compounded yearly, is approximately $\frac{70}{i}$ for small values of i. Use the differential formula $f(1 + dx) \approx f(1) + dy$ for the function $\ln(x)$ to verify this rule.