Solutions to Supplementary Questions for HP Chapter 12

1. 1) $\frac{d}{dx}(x^8 \ln x) = 8x^7 \ln x + \frac{x^8}{x} = 8x^7 \ln x + x^7$

2) $\frac{d^2}{dx^2}(x^8 \ln x) = \frac{d}{dx}(8x^7 \ln x + x^7) = \left(8 \cdot 7x^6 \ln x + \frac{8x^7}{x}\right) + 7x^6$
$= 8 \cdot 7x^6 \ln x + 8x^6 + 7x^6$

Now note that the power of x in all terms are the same, namely $8-k$ when we calculate $\frac{d^k}{dx^k}$, so at the 8^{th} derivative, the power of x is zero in all terms. Also note that all terms except for the first term are only a constant times a power of x, so at the eighth derivative, all terms other than the first term become constant. Hence at the 9^{th} derivative, the only term that is nonzero is the first term that includes $\ln x$.

8) $\frac{d^8}{dx^8}(x^8 \ln x) = 8 \cdot 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 \cdot x^0 \ln x +$ constants

9) $\frac{d^9}{dx^9}(x^8 \ln x) = \frac{8 \cdot 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 \cdot}{x} \quad (= \frac{8!}{x})$

2.

$$\begin{aligned}
&\frac{d}{dx}[\ln(\ln(\ln(\ln x)))] \\
&= \frac{1}{\ln(\ln(\ln x))}\left[\frac{d}{dx}\ln(\ln(\ln x))\right] \\
&= \frac{1}{\ln(\ln(\ln x))}\left(\frac{1}{\ln(\ln x)}\frac{d}{dx}(\ln(\ln x))\right) \\
&= \frac{1}{\ln(\ln(\ln x))}\left(\frac{1}{\ln(\ln x)}\right)\frac{1}{\ln x}\frac{d}{dx}(\ln x) \\
&= \frac{1}{\ln(\ln(\ln x))}\left(\frac{1}{\ln(\ln x)}\right)\frac{1}{\ln x}\frac{1}{x}
\end{aligned}$$

3. (i) $\frac{d}{dx}x^{(a^a)} = a^a x^{(a^a-1)}$

(ii) Using the Chain Rule, $\frac{d}{dx}\left(a^{f(x)}\right) = a^{f(x)} \ln a(f'(x))$, and hence, $\frac{d}{dx}a^{(x^a)} = a^{(x^a)} \ln a(ax^{a-1})$

(iii) Similarly, $\frac{d}{dx}a^{(a^x)} = a^{(a^x)} \ln a(\frac{d}{dx}a^x) = a^{(a^x)} \ln a(a^x \ln a) = a^{(a^x)}a^x(\ln a)^2$.

So $\frac{d}{dx}\left(x^{(a^a)} + a^{(x^a)} + a^{(a^x)}\right) = a^a x^{(a^a-1)} + a \ln a\left(a^{(x^a)}\right)x^{a-1} + a^{(a^x)}a^x(\ln a)^2$.

4. (a)

$$\begin{aligned} f'(x) &= \lim_{h\to 0} \frac{f(x+h)-f(x)}{x} \\ &= \lim_{h\to 0} \frac{f(x)f(h)-f(x)}{x} \\ &= \lim_{h\to 0} f(x)\left[\frac{f(h)-1}{h}\right] \\ &= f(x)\lim_{h\to 0} \frac{f(h)-f(0)}{h} \\ &= f(x)f'(0) \\ &= f(x) \end{aligned}$$

(b) We have, for $f(x) = e^x$,

$f'(0) = \lim_{h\to 0} \frac{e^h-1}{h} = 1.$

$f(0) = e^0 = 1$

$f(x+z) = e^{x+z} = e^x e^z = f(x)f(z)$ for all x and z.

Applying (a) shows $f(x) = f'(x)$; i.e., $\frac{d}{dx}e^x = e^x$.

5. Let $g(x) = f(x)e^{-cx}$. By the product rule and chain rule we have

$$\begin{aligned} g'(x) &= f'(x)e^{-cx} - f(x)ce^{-cx} \\ &= cf(x)e^{-cx} - cf(x)e^{-cx} \\ &= 0 \end{aligned}$$

Since the derivative of g is zero, and g is defined for all x, we must have g equal to a constant K, $g(x) = K$, so $K = f(x)e^{-cx}$ and multiplying both sides by e^{cx} gives $f(x) = Ke^{cx}$.

6. (a)

$f(x) = \ln(x)$	$g(x) = ax^2 + bx + c$
$f'(x) = \frac{1}{x}$	$g'(x) = 2ax + b$
$f''(x) = -\frac{1}{x^2}$	$g''(x) = 2a$

so we have

$$\begin{aligned} g(1) = f(1) &\quad\Rightarrow\quad a+b+c=0 \\ g'(1) = f'(1) &\quad\Rightarrow\quad 2a+b=1 \\ g''(1) = f''(1) &\quad\Rightarrow\quad 2a=-1 \end{aligned}$$

Solving, we see that $a = -\frac{1}{2}$, $b = 2$, $c = -\frac{3}{2}$, so $g(x) = -\frac{1}{2}x^2 + 2x - \frac{3}{2}$.

(b)

$$f(x) = f'(x) = f''(x) = e^x$$

$$g(x) = ax^2 + bx + c$$

$$g'(x) = 2ax + b$$

$$g''(x) = 2a$$

so we have

$$\begin{aligned} g(0) &= f(0) \quad \Rightarrow \quad c = 1 \\ g'(0) &= f'(0) \quad \Rightarrow \quad b = 1 \\ g''(0) &= f''(0) \quad \Rightarrow \quad 2a = 1 \quad \Rightarrow \quad a = \frac{1}{2} \\ \text{so } g(x) &= \frac{1}{2}x^2 + x + 1 \end{aligned}$$

7. (a) The principal in the bank account at the beginning of the year will be $P(t)$, and will earn 5% interest compounded annually for a period of $20 - t$ years.

(b)

$$\begin{aligned} B'(t) &= P'(t)(1.05)^{20-t} - P(t)\ln(1.05)(1.05)^{20-t} \\ \text{so } B'(10) &= 5000(1.05)^{10} - 150000\ln(1.05)(1.05)^{10} \\ &= -3776.63 \end{aligned}$$

$$\begin{aligned} B(11) &= B(10) + B'(10)(1) \\ &= P(10)(1.05)^{10} - 3776.63 \\ &= 150000(1.05)^{10} - 3776.63 \\ &= 240557.56 \end{aligned}$$

(c) $B(t) = P(t)e^{0.05(20-t)}$

8. (a)

$$\frac{dq}{dp} = abp^{b-1} \quad \text{so} \quad \eta = \frac{p}{q}\frac{dq}{dp} = \frac{p}{q}abp^{b-1} = \frac{p}{ap^b}abp^{b-1} = b$$

which is a constant.

(b) Using the product rule, we have

$$\begin{aligned} \frac{p}{q_1q_2}\frac{d(q_1q_2)}{dp} &= \frac{p}{q_1q_2}(q_1\frac{dq_2}{dp} + q_2\frac{dq_1}{dp}) \\ &= \frac{p}{q_2}\frac{dq_2}{dp} + \frac{p}{q_1}\frac{dq_1}{dp} \end{aligned}$$

(c) Write $q_1 = 3p^{-5}$, $q_2 = \frac{p-5}{6}$. q_1 has point elasticity -5 by (a). If $q_2 = \frac{p-5}{6}$, $p = 6q_2 + 5$ so q_2 has point elasticity $\frac{p}{p-5}$ which is $-\frac{2}{3}$ when $p = 2$.

From (b), we know that since $q = q_1 q_2$, the point elasticity of q when $p = 2$ is $-5 + \frac{-2}{3} = \frac{-17}{3}$.

9. By the chain rule, $\frac{dy}{dx} = \frac{dy}{dp}\frac{dp}{dx}$. For $\frac{dy}{dp}$, we have

$$\frac{dy}{dp} = \frac{d(\ln f(p))}{dp} = \frac{1}{f(p)} f'(p) = \frac{1}{q}\frac{dq}{dp}.$$

For $\frac{dp}{dx}$ we have $x = \ln p$, $p = e^x$, $\frac{dp}{dx} = e^x = p$. Therefore,

$$\frac{dy}{dx} = \frac{dy}{dp}\frac{dp}{dx} = \left(\frac{1}{q}\frac{dq}{dp}\right) p = \frac{p}{q}\frac{dq}{dp} = \eta$$

10. (a) Taking natural logarithms, we have

$$\ln q = a(\ln p) - b(p + c)$$

and differentiating with respect to p, we have

$$\frac{1}{q}\frac{dq}{dp} = \frac{a}{p} - b$$

and

$$\frac{dq}{dp} = \left(\frac{a}{p} - b\right)\left(p^a e^{-b(p+c)}\right)$$

Since $p > \frac{a}{b}$, $\frac{a}{p} < b$ which shows that for all $p > \frac{a}{b}$, $\frac{dq}{dp} < 0$. This means that the demand decreases as the price increases or equivalently, that the demand increases as the price decreases.

(b) Using the equation $\eta = \frac{p}{q}\frac{dq}{dp}$ where η is the point elasticity of demand, we have

$$\begin{aligned}
\eta &= \frac{p}{q}\left(\frac{a}{p} - b\right)\left(p^a e^{-b(p+c)}\right) \\
&= \frac{p}{q}\left(\frac{a}{p} - b\right) q \\
&= a - bp
\end{aligned}$$

11.

$$x^n = e^{\ln(x^n)} = e^{n \ln x}$$

$$\Rightarrow \frac{d}{dx}x^n = \frac{d}{dx}e^{n\ln x} = e^{n \ln x}\left(\frac{d}{dx} n \ln x\right) = x^n\left(\frac{n}{x}\right) = nx^{n-1}$$

OR

$$y = x^n \quad \Rightarrow \quad \ln y = n \ln x \quad \Rightarrow \quad \frac{y'}{y} = \frac{n}{x} \quad \Rightarrow \quad y' = x^n\left(\frac{n}{x}\right) = nx^{n-1}$$

12. Using implicit differentiation with respect to x,

$$\frac{1}{2\sqrt{x}} + \frac{1}{2\sqrt{y}}\frac{dy}{dx} = 0$$
$$\frac{dy}{dx} = -\frac{2\sqrt{y}}{2\sqrt{x}} = -\sqrt{\frac{y}{x}} \qquad (x \neq 0)$$

An arbitrary point on the curve is $(x_0, (\sqrt{c} - \sqrt{x_0})^2)$, so the tangent line has the following equation (note that $x_0 \neq 0$).

$$y - (\sqrt{c} - \sqrt{x_0})^2 = (x - x_0)\left(-\sqrt{\frac{(\sqrt{c} - \sqrt{x_0})^2}{x_0}}\right)$$
$$y - (\sqrt{c} - \sqrt{x_0})^2 = \frac{(x - x_0)(\sqrt{x_0} - \sqrt{c})}{\sqrt{x_0}}$$

When $x = 0$, then

$$\begin{aligned} y &= -\frac{x_0}{\sqrt{x_0}}(\sqrt{x_0} - \sqrt{c}) + (\sqrt{c} - \sqrt{x_0})^2 \\ &= \sqrt{x_0}(\sqrt{c} - \sqrt{x_0}) + (\sqrt{c} - \sqrt{x_0})^2 \\ &= (\sqrt{c} - \sqrt{x_0})(\sqrt{c} - \sqrt{x_0} + \sqrt{x_0}) \\ &= \sqrt{c}(\sqrt{c} - \sqrt{x_0}) \end{aligned}$$

So the y-intercept is $\sqrt{c}(\sqrt{c} - \sqrt{x_0}) = c - \sqrt{cx_0}$.

When $y = 0$ (note that $y_0 \neq 0$),

$$x - x_0 = -\frac{(\sqrt{c} - \sqrt{x_0})^2\sqrt{x_0}}{\sqrt{x_0} - \sqrt{c}} = (\sqrt{c} - \sqrt{x_0})\sqrt{x_0}$$
$$x = \sqrt{x_0}(\sqrt{c} - \sqrt{x_0}) + x_0 = \sqrt{x_0 c} - x_0 + x_0 = \sqrt{x_0 c}$$

so the x intercept is $\sqrt{x_0 c}$. The x-intercept and y-intercept have sum $\sqrt{x_0 c} + c - \sqrt{cx_0} = c$.

13. (a) $f(g(x)) = x$. Differentiating, we have $f'(g(x))g'(x) = 1$, so $g'(x) = \frac{1}{f'(g(x))}$.

(b) $g'(x) = \frac{1}{f'(g(x))} = \frac{1}{\left(\frac{1}{e^x}\right)} = e^x$ $f'(x) = \frac{1}{g'(f(x))} = \frac{1}{e^{\ln x}} = \frac{1}{x}$

14. Differentiating implicitly with respect to x, we have

$$2x - x\frac{dy}{dx} - y + 2y\frac{dy}{dx} = 0$$

so

$$\frac{dy}{dx} = \frac{y - 2x}{2y - x}$$

When $y = 0$,

$$\frac{dy}{dx} = \frac{-2x}{-x} = 2$$

and $x^2 - xy + y^2 = 9$ so when $y = 0$, $x^2 = 9$, $x = \pm 3$. The tangent lines have the equations

$$y = 2(x - 3)$$
$$y = 2(x + 3)$$

Since they have the same slope (2) they are parallel.

15. Differentiating the equation of the circle implicitly with respect to x,

$$\begin{aligned} 2(x-a) + 2(y-b)\frac{dy}{dx} &= 0 \\ (x-a) + (y-b)\frac{dy}{dx} &= 0 \qquad (1) \\ \frac{dy}{dx} &= -\frac{(x-a)}{(y-b)} \end{aligned}$$

Differentiating equation (1) implicitly with respect to x,

$$\begin{aligned} 1 + \left(\frac{dy}{dx}\right)^2 + (y-b)\frac{d^2y}{dx^2} &= 0 \\ 1 + \left(\frac{dy}{dx}\right)^2 &= -(y-b)\frac{d^2y}{dx^2} \qquad (2) \\ \left[1 + \left(\frac{x-a}{y-b}\right)^2\right]\left(-\frac{1}{(y-b)}\right) &= \frac{d^2y}{dx^2} \end{aligned}$$

Taking both sides of equation (2) to the power $\frac{3}{2}$, we have

$$\begin{aligned} \left[1 + \left(\frac{dy}{dx}\right)^2\right]^{\frac{3}{2}} &= -(y-b)\frac{d^2y}{dx^2}\sqrt{-(y-b)\frac{d^2y}{dx^2}} \\ &= -(y-b)\frac{d^2y}{dx^2}\sqrt{1 + \left(\frac{x-a}{y-b}\right)^2} \\ &= -(y-b)\frac{d^2y}{dx^2}\sqrt{\frac{(x-a)^2 + (y-b)^2}{(y-b)^2}} \\ &= -(y-b)\frac{d^2y}{dx^2}\sqrt{\frac{r^2}{(y-b)^2}} \\ &= \pm r\frac{d^2y}{dx^2} \end{aligned}$$

Therefore, $\frac{1}{r} = \pm\frac{\frac{d^2y}{dx^2}}{\left[1+\left(\frac{dy}{dx}\right)^2\right]^{\frac{3}{2}}}$; or $\frac{1}{r} = \left|\frac{\frac{d^2y}{dx^2}}{\left[1+\left(\frac{dy}{dx}\right)^2\right]^{\frac{3}{2}}}\right|$

16. Differentiating implicitly the first equation with respect to z,

$$\begin{aligned} y + z\frac{dy}{dz} + y3z^2 + z^3\frac{dy}{dz} &= 0 \\ (z+z^3)\frac{dy}{dz} &= -y(1+3z^2) \\ \frac{dy}{dz} &= -\frac{y(1+3z^2)}{z+z^3} \end{aligned}$$

and also differentiating the second question with respect to x,

$$\begin{aligned} 2xz + x^2\frac{dz}{dx} + 3z^2 + 6xz\frac{dz}{dx} &= 6x^2y + 2x^3\frac{dy}{dx} \\ (x^2+6xz)\frac{dz}{dx} &= 6x^2y - 2xz - 3z^2 + 2x^3\frac{dy}{dx} \\ \frac{dz}{dx} &= \frac{6x^2y - 2xz - 3z^2 + 2x^3\frac{dy}{dx}}{x^2+6xz} \end{aligned}$$

By the chain rule, $\frac{dy}{dx} = \frac{dy}{dz}\frac{dz}{dx}$, so

$$\frac{dy}{dx} = -y\frac{(1+3z^2)}{z+z^3}\cdot\frac{6x^2y - 2xz - 3z^2 + 2x^3\frac{dy}{dx}}{x^2+6xz}$$

Solving for $\frac{dy}{dx}$ we have

$$\begin{aligned} (z+z^3)(x^2+6xz)\frac{dy}{dx} &= -y(1+3z^2)(6x^2y - 2xz - 3z^2 + 2x^3\frac{dy}{dx}) \\ \frac{(z+z^3)(x^2+6xz)}{-y(1+3z^2)}\frac{dy}{dx} &= 6x^2y - 2xz - 3z^2 + 2x^3\frac{dy}{dx} \\ \left[2x^3 + \frac{(z+z^3)(x^2+6xz)}{y(1+3z^2)}\right]\frac{dy}{dx} &= 2xz + 3z^2 - 6x^2y \\ \frac{dy}{dx} &= \frac{(2xz+3z^2-6x^2y)y(1+3z^2)}{y(1+3z^2)2x^3 + (z+z^3)(x^2+6xz)} \end{aligned}$$

17. Set $y = \frac{e^x\sqrt{x^5+2}}{(x+1)^4(x^2+3)^2}$ Then

$$\begin{aligned} \ln y &= \ln\left(\frac{e^x\sqrt{x^5+2}}{(x+1)^4(x^2+3)^2}\right) \\ &= \ln(e^x) + \ln\sqrt{x^5+2} - \ln(x+1)^4 - \ln(x^2+3)^2 \\ &= x + \frac{1}{2}\ln(x^5+2) - 4\ln(x+1) - 2\ln(x^2+3) \end{aligned}$$

Differentiating both sides with respect to x gives

$$\begin{aligned}
\frac{d\ln y}{dx} &= \frac{d}{dx}\left[x + \frac{1}{2}\ln(x^5+2) - 4\ln(x+1) - 2\ln(x^2+3)\right] \\
\Rightarrow \frac{y'}{y} &= 1 + \frac{\frac{d}{dx}(x^5+2)}{2(x^5+2)} - \frac{4(\frac{d}{dx}(x+1))}{x+1} - \frac{2(\frac{d}{dx}(x^2+3))}{x^2+3} \\
\Rightarrow \frac{y'}{y} &= 1 + \frac{5x^4}{2(x^5+2)} - \frac{4}{x+1} - \frac{4x}{x^2+3}
\end{aligned}$$

Substituting back in $y = \frac{e^x\sqrt{x^5+2}}{(x+1)^4(x^2+3)^2}$ and solving for y' gives

$$y' = \frac{e^x\sqrt{x^5+2}}{(x+1)^4(x^2+3)^2}\left(1 + \frac{5x^4}{2(x^5+2)} - \frac{4}{x+1} - \frac{4x}{x^2+3}\right)$$

18. (a) First, set $u = x^x$. Then

$$\begin{aligned}
&\ln u = \ln(x^x) = x\ln x \\
&\Rightarrow \frac{u'}{u} = \ln x + \frac{x}{x} = \ln x + 1 \\
&\Rightarrow u' = x^x(\ln x + 1)
\end{aligned}$$

Now, set $y = x^{(x^x)}$. Then $\ln y = \ln x^{(x^x)} = x^x \ln x$. Taking the derivative of both sides gives

$$\begin{aligned}
&\frac{y'}{y} = x^x(\ln x + 1)\ln x + \frac{x^x}{x} \\
&\Rightarrow y' = x^{(x^x)}\left[x^x((\ln x)^2 + \ln x) + x^{x-1}\right]
\end{aligned}$$

(b) $y = x^{\ln x} \Rightarrow \ln y = \ln(x^{\ln x}) = \ln x(\ln x) = (\ln x)^2$. Taking the derivative of both sides gives

$$\begin{aligned}
&\frac{y'}{y} = 2(\ln x)\left[\frac{d}{dx}(\ln x)\right] = \frac{2(\ln x)}{x} \\
&\Rightarrow y' = x^{\ln x}\left[\frac{2\ln x}{x}\right] = 2\ln x(x^{(\ln x)-1})
\end{aligned}$$

(c) $y = (\ln x)^x \Rightarrow \ln y = \ln(\ln x)^x = x\ln(\ln x)$. Taking the derivative of both sides give

$$\begin{aligned}
\frac{y'}{y} &= \ln(\ln x) + \frac{x}{\ln x}\left[\frac{d}{dx}(\ln x)\right] \\
&= \ln(\ln x) + \frac{x}{x\ln x} = \ln(\ln x) + \frac{1}{\ln(x)} \\
\Rightarrow y' &= (\ln x)^x\left[\ln(\ln x) + \frac{1}{\ln x}\right]
\end{aligned}$$

(d) $y = f(x)^{g(x)} \Rightarrow \ln y = \ln(f(x)^{g(x)}) = g(x)\ln(f(x))$. Differentiating both sides gives

$$\begin{aligned}\frac{y'}{y} &= g'(x)\ln(f(x)) + \frac{g(x)f'(x)}{f(x)}\\ \Rightarrow y' &= f(x)^{g(x)}\left[g'(x)\ln(f(x)) + \frac{g(x)f'(x)}{f(x)}\right]\end{aligned}$$

19. **(a)** We wish to solve $f(x) = x^k - a = 0$, $f'(x) = kx^{k-1}$, so the formula $x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$ gives

$$\begin{aligned}x_{n+1} &= x_n - \frac{x_n^k - a}{kx_n^{k-1}}\\ &= x_n - \frac{x_n^k}{kx_n^{k-1}} + \frac{a}{kx_n^{k-1}}\\ &= \frac{1}{k}\left[kx_n - x_n + \frac{a}{(x_n)^{k-1}}\right]\\ &= \frac{1}{k}\left[(k-1)x_n + \frac{a}{(x_n)^{k-1}}\right]\end{aligned}$$

(b) Here $k = 10$, $a = 100$. Since $1^{10} = 1$, $2^{10} = 1024$, we let $x_0 = 1.5$. The formula then gives

$$\begin{aligned}x_1 &= 1.610122949\\ x_2 &= 1.58659981\\ x_3 &= 1.58490143\\ x_4 &= 1.584893193\\ x_5 &= 1.584893193\end{aligned}$$

Having obtained $x_4 = x_5$ up to nine digits of accuracy, we may certainly conclude that 1.58489 is accurate to five digits.

20. The equation becomes

$$\begin{aligned}2000[(1+r)^{30} - 1] &= 8000[1 - (1+r)^{-20}]\\ (1+r)^{30} - 1 &= 4 - 4(1+r)^{-20}]\end{aligned}$$

or $f(r) = (1+r)^{30} + 4(1+r)^{-20} - 5 = 0$ with $f'(r) = 30(1+r)^{29} - 80(1+r)^{-21}$. Newton's method gives

$$r_{n+1} = r_n - \frac{(1+r_n)^{30} + 4(1+r_n)^{-20} - 5}{30(1+r_n)^{29} - 80(1+r_n)^{-21}}$$

With an initial guess of $r_=0.05$, the iterates are

$$r_0 = 0.05$$
$$r_1 = 0.041247105$$
$$r_2 = 0.03894101$$
$$r_3 = 0.038779215$$
$$r_4 = 0.038778432$$

r_3 and r_4 are the same to five decimal digits, so we may conclude r_4 is accurate to four decimal digits, indicating the interest rate was 3.88% (rounded off).

21. Plugging in the values for P, L, p, g, we have $160z = e^{70z} - e^{-70z}$ where $z = \frac{2.4525}{T}$. Therefore we wish to solve $f(z) = e^{70z} - e^{-70z} - 160z = 0$. $f'(z) = 70e^{70z} + 70e^{-70z} - 160$. The Newton's method formula gives

$$x_{n+1} = x_n - \frac{e^{70x_n} - e^{-70x_n} - 160x_n}{70e^{70x_n} + 70e^{-70x_n} - 160}$$

We then have

$$\begin{array}{ll} x_0 = 0.05 & T_0 = 49.05 \\ x_1 = 0.038387507 & T_1 = 63.8880 \\ x_2 = 0.028674915 & T_2 = 85.5277 \\ x_3 = 0.021330325 & T_3 = 114.9771 \\ x_4 = 0.016467949 & T_4 = 148.9256 \\ x_5 = 0.01388673 & T_5 = 176.6075 \\ x_6 = 0.01304748 & T_6 = 187.9673 \\ x_7 = 0.012958021 & T_7 = 189.2650 \\ x_8 = 0.012957042 & T_8 = 189.2793 \\ x_9 = 0.01295704 & T_9 = 189.2793 \end{array}$$

Since $T_8 = T_9$ up to four digits of accuracy, we may certainly conclude $T = 189.3$ is accurate to one digit (rounded off).

22. Since $\lim_{x\to 0^+} f(x) = \lim_{x\to 0^-} f(x) = 0 = f(0)$, f is continuous at $x = 0$. Everywhere else f is a polynomial, so f is continuous.

We use the definition of derivative to determine if f has a derivative at $x = 0$. Let $\operatorname{sgn}(x)$ be the function defined by $\operatorname{sgn}(x) = \begin{cases} 1; & x > 0 \\ 0; & x = 0 \\ -1; & x < 0 \end{cases}$. Then

$$\begin{aligned} \lim_{h\to 0} \frac{f(0+h) - f(0)}{h} &= \lim_{h\to 0} \frac{\operatorname{sgn}(h)\frac{1}{2}(h)^2}{h} - 0 \\ &= \lim_{h\to 0} \frac{1}{2} \operatorname{sgn}(h) h \\ &= 0 \end{aligned}$$

so $f'(0) = 0$.

Everywhere else f has the usual derivative of a polynomial, so we may write

$$f'(x) = \begin{cases} x; & x \geq 0 \\ -x; & x < 0 \end{cases}$$
$$= |x|$$

As shown in the text, example 2, page 637, $f'(x)$ is not differentiable at $x = 0$ so $f(x)$ does not have a second derivative at $x = 0$ but does have a derivative at $x = 0$.

23. **(a)** Each derivative reduces the powers of x by 1, up until $x^0 = 1$, whose derivative is 0. Taking ten derivatives will therefore give 0.

(b) By the above argument, we need only consider $3x^9 + 5x^8$. The eighth derivative is

$$3 \cdot 9 \cdot 8 \cdot 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot x + 5 \cdot 8 \cdot 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2$$
$$= 3(9!)x + 5(8!) = 1088640x + 201600$$

(c) $g(t) = 3t^{27} + 5t^{24} + t^{18} + 5t^{15} - 4t^9 - t^3 + 1$. By the same argument as (a), the 28^{th} derivative will be zero.

(d) We need only consider $3t^{27}$ since the other terms will have 26^{th} derivatives of 0. The 26^{th} derivative is $3(27!)t$.

24. The velocity of the particle is $x'(t) = Ace^{ct} - Bce^{-ct}$ and the acceleration is $x''(t) = Ac^2e^{ct} + Bc^2e^{-ct} = c^2(Ae^{ct} + Be^{-ct}) = c^2x(t)$ so the acceleration is given by the constant c^2 multiplied with the position, i.e., the acceleration is proportional to the position.