Supplementary Questions for HP Chapter 12

1. Find $\dfrac{d^9}{dx^9}(x^8 \ln x)$.

2. Find $\dfrac{d}{dx}[\ln(\ln(\ln(\ln x)))]$.

3. Find $\dfrac{d}{dx}(x^{(a^a)} + a^{(x^a)} + a^{(a^x)})$ where a is a constant.

4. **(a)** Let $f(x)$ be a function such that $f(x+z) = f(x)f(z)$ for all x and z, and $f(0) = f'(0) = 1$. Prove that $f'(x) = f(x)$, using the definition of $f'(x)$.

(b) Suppose you know $\lim_{h\to 0} \frac{e^h - 1}{h} = 1$. Show $\frac{d}{dx}e^x = e^x$.

5. Let f be a function defined everywhere on the real axis, with a derivative f' which satisfies the equation $f'(x) = cf(x)$ for all x, where c is a constant. Prove that there is a K such that $f(x) = Ke^{cx}$ for all x. Hint: Let $g(x) = f(x)e^{-cx}$ and consider $g'(x)$.

6. **(a)** Find the quadratic function $g(x) = ax^2+bx+c$ such that $g(1) = f(1)$, $g'(1) = f'(1)$, $g''(1) = f''(1)$ where $f(x) = \ln x$.

(b) Find the quadratic function $g(x) = ax^2 + bx + c$ such that $g(0) = f(0)$, $g'(0) = f'(0)$, $g''(0) = f''(0)$ where $f(x) = e^x$.

7. A museum has decided to sell a painting and invest the proceeds. The price the painting will fetch changes with time, given by the function $P(t)$. If the painting is sold between 1996 and 2016 and invested in a bank account earning 5% annual interest compounded once a year, the balance $B(t)$ in the account in the year 2016 is given by $B(t) = P(t)(1.05)^{20-t}$ where t is the number of years after 1996 it is sold.

(a) Explain why $B(t)$ is given by this formula.

(b) Find $B'(10)$ given that $P(10) = 150000$ and $P'(10) = 5000$. Estimate $B(11)$ using $B'(10)$.

(c) What would the formula be if interest is compounded continuously in the bank account?

8. **(a)** consider the demand equation $q = ap^b$ where p is price, q is demand, $a > 0$, $b < 0$ are constants. Show that the point elasticity of demand n is constant.

(b) Given two demand functions $q_1(p)$ and $q_2(p)$, show that

$$\frac{p}{q_1 q_2}\frac{d(q_1 q_2)}{dp} = \frac{p}{q_2}\frac{dq_2}{dp} + \frac{p}{q_1}\frac{dq_1}{dp}$$

This shows that the point elasticity of the product of two demand functions is the sum of the elasticities of the two functions.

(c) Use (a), (b) and the rule that the linear equation $p = aq + b$ has point elasticity $\frac{p}{p-b}$ (shown in the text) to compute the point elasticity at $p = 2$ where $q = (3p^{-5})\left(\frac{p-5}{6}\right)$.

9. Given a demand function, $q = f(p)$, show that the point elasticity of demand η is given by $\frac{dy}{dx} = \eta$ where $y = \ln f$, $x = \ln p$ by considering y as a function of x, and using the chain rule.

10. If a company sells a certain commodity at a price p, the market demands $q = p^a e^{-b(p+c)}$ items per week, where $p > \frac{a}{b}$ and a, b, c are positive constants.

(a) Show that the demand increases as the price decreases.

(b) Calculate the point elasticity of demand, in terms of p.

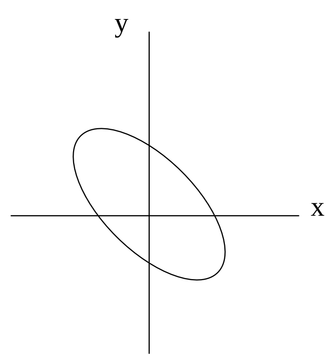
11. In HP, it was shown that $\frac{d}{dx}x^n = nx^{n-1}$. But this was only for positive integers n. Assuming that $x > 0$, let n be any real number. Prove that $\frac{d}{dx}x^n = nx^{n-1}$.

12. Show that the sum of the x and y intercepts of any tangent line to the curve $\sqrt{x}+\sqrt{y} = \sqrt{c}$ is equal to c. $(x \neq 0, y \neq 0)$

13. (a) The functions f and g are inverse functions of each other if $f(g(x)) = x$ and $g(f(x)) = x$ for all x. Suppose f is differentiable everywhere and $f'(x) \neq 0$ for all x. Differentiate both sides of $f(g(x)) = x$, the left side using the chain rule, to obtain a formula for $g'(x)$.

(b) $f(x) = \ln x$ and $g(x) = e^x$ are inverse functions. Use (a) to solve for $g'(x)$ if you know $f'(x) = \frac{1}{x}$. Solve for $f'(x)$ if you know $g'(x) = e^x$.

14. The graph of the equation $x^2 - xy + y^2 = 9$ is a rotated ellipse as shown in the graph.



Find the tangent lines to this curve at the two points where it intersects the x-axis, and show that these lines are parallel.

15. Show that for any circle $(x-a)^2 + (y-b)^2 = r^2$,

$$\left| \frac{\frac{d^2y}{dx^2}}{\left[1 + \left(\frac{dy}{dx}\right)^2\right]^{\frac{3}{2}}} \right| = \frac{1}{r}$$

16. Given that the equations

$$\begin{aligned} yz + yz^3 &= 3 \\ x^2 z + 3xz^2 &= 2x^3 y + 1 \end{aligned}$$

define y as a function of z and z as a function of x, find $\frac{dy}{dx}$ in terms of x, y, z.

17. Find $\frac{d}{dx}\left[\frac{e^x\sqrt{x^5+2}}{(x+1)^4(x^2+3)^2}\right]$.

18. Find the derivatives of

(a) $y = x^{(x^x)}$

(b) $y = x^{\ln x}$

(c) $y = (\ln x)^x$

(d) $f(x)^{g(x)}$

19. **(a)** Show that Newton's method yields the iteration

$$x_{n+1} = \frac{1}{k}\left[(k-1)x_n + \frac{a}{(x_n)^{k-1}}\right]$$

for approximating the k^{th} root of the positive number a, from the equation $x^k - a = 0$.

(b) Use this iteration to find $\sqrt[10]{100}$ accurate to five decimal places.

20. Consider the following annuity problem. An amount of P_1 dollars is put into an account at the beginning of years $1, 2, 3, \ldots, N_1$. It is compounded annually at an interest rate r. At the beginning of years $N_1+1, N_1+2, \ldots, N_1+N_2$, a payment of P_2 dollars is removed from the account. After the last payment, the account balance is exactly zero.

The deposits into the account are an annuity that will result in an account balance of $P_1\left[\frac{(1+r)^{N_1}-1}{r}\right]$ at the end of year N_1.

The payments from the account are an annuity that will have a present value of $P_2\left[\frac{1-(1+r)^{-N_2}}{r}\right]$ at the beginning of year N_1+1 (which is the end of year N_1).

Since the account balance is zero after the last payment, we have

$$P_1\left[\frac{(1+r)^{N_1}-1}{r}\right]=P_2\left[\frac{1-(1+r)^{-N_2}}{r}\right]$$

or

$$P_1\left[(1+r)^{N_1}-1\right]=P_2\left[1-(1+r)^{-N_2}\right]$$

Use Newton's method to solve for r when $N_1 = 30$, $N_2 = 20$, $P_1 = 2000$, $P_2 = 8000$. (Four decimal digits of accuracy.)

21. A uniform hydro cable $P = 80$m long, with mass per unit length $p = 0.5$ kg/m is hung from two supports at the same level $L = 70$m apart.

The tension T in the cable at its lowest point must satisfy the equation $\frac{pgP}{T} = e^{\frac{pgL}{2T}} - e^{-\frac{pgL}{2T}}$ where $g = 9.81$. If we set $z = \frac{pg}{2T}$, then z must satisfy $2Pz = e^{Lz} - e^{-Lz}$. . Find T correct to one decimal by solving this equation for z using Newton's method. (Use an initial guess of 0.05).

22. Consider the function

$$f(x)=\begin{cases}\frac{1}{2}x^2; & x\geq 0\\ -\frac{1}{2}x^2; & x<0\end{cases}$$

Is f continuous? Does f have a derivative at $x = 0$? Does f have a second derivative at $x = 0$?

23. Let $f(x) = 3x^9 + 5x^8 + x^6 + 5x^5 - 4x^3 - x + 1$.

(a) Find the 10^{th} derivative of $f(x)$.

(b) Find the 8^{th} derivative of $f(x)$.

(c) Let $g(t) = f(t^3)$. Find the 28^{th} derivative of g, with respect to t.

(d) Find the 26^{th} derivative of g, with respect to t.

24. Note that velocity is the rate of change of position with respect to time, and that acceleration is the rate of change of velocity with respect to time.

If a particle moves on a coordinate line with its position at time t given by $x(t) = Ae^{ct} + Be^{-ct}$ where A, B, c are constants, show that the particle's acceleration is proportional to its position.