Solutions to Supplementary Questions for HP Chapter 11

1. The curve has derivative $3x^2$. Let (x_0, x_0^3) be any point on the curve. Then the tangent line at this point has slope $3x_0^2$, so if $(1, 0)$ is on this line, then the line has equation

$$y = 3x_0^2(x - 1)$$

Since (x_0, x_0^3) is also on this line, we must have

$$\begin{aligned} x_0^3 &= 3x_0^2(x_0 - 1) \\ x_0^3 &= 3x_0^3 - 3x_0^2 \\ 2x_0^3 - 3x_0^2 &= 0 \\ x_0^2(2x_0 - 3) &= 0 \end{aligned}$$

so we may conclude $x_0 = 0$ or $x_0 = \frac{3}{2}$. From the equation $y = 3x_0^2(x - 1)$, we see the two straight lines are $y = 0$ and $y = \frac{27}{4}(x - 1)$.

2. **(a)** The curve $y = x^2$ has derivative $2x$, so the tangent line at the point $(2, 4)$ has slope 4, and therefore the normal line has slope $-\frac{1}{4}$ (the negative reciprocal). The equation of the normal line is given by

$$\begin{aligned} y - 4 &= -\frac{1}{4}(x - 2) \\ \Rightarrow y &= -\frac{1}{4}x + \frac{9}{2} \end{aligned}$$

(b) Much the same as above, the tangent line at the point (x_0, x_0^2) has slope $2x_0$ and the normal line has slope $-\frac{1}{2x_0}$, so the equation of the normal line is

$$\begin{aligned} y - x_0^2 &= -\frac{1}{2x_0}(x - x_0) \\ \Rightarrow y &= -\frac{1}{2x_0}x + x_0^2 + \frac{1}{2} \end{aligned}$$

(c) From (b), we know the normal line of the curve $y = x^2$ at an arbitrary point (x_0, x_0^2) has equation

$$y = -\frac{1}{2x_0}x + x_0^2 + \frac{1}{2}$$

If this line passes through $(2, \frac{1}{2})$, then the equation must be satisfied by this point, so

$$\frac{1}{2} = -\frac{1}{2x_0}(2) + x_0^2 + \frac{1}{2}$$

Solving for x_0,

$$\begin{aligned}\frac{1}{2} &= -\frac{1}{x_0} + x_0^2 + \frac{1}{2} \\ \Rightarrow x_0^2 &- \frac{1}{x_0} = 0 \\ \Rightarrow x_0^3 &- 1 = 0\end{aligned}$$

The solution is $x_0 = 1$, so the point $(1,1)$ on the curve $y = x^2$ is the only point where the normal line passes through the point $(2, \frac{1}{2})$.

3. (a) The curve has derivative $5x^4 + 2$. Since $x^4 \geq 0$ for all x, $5x^4 + 2 \geq 2$ for all x, and since the derivative is never equal to zero, there is no horizontal tangent line.

(b) The curve has derivative $2x + a$. If the point $(0,1)$ is on the curve, we must have $1 = 0^2 + a(0) + b$ so $b = 1$. We must also have the derivative equal to zero at $(0,1)$, so $0 = 2(0) + a$ which shows $a = 0$. Therefore the curve has the equation $y = x^2 + 1$.

4. We can write

$$|x|^n = \begin{cases} x^n; & x \geq 0 \\ (-x)^n; & x < 0 \end{cases}$$

so for $x \neq 0$ we have

$$\begin{aligned}\frac{d}{dx}|x|^n &= \begin{cases} nx^{n-1}; & x > 0 \\ -n(-x)^{n-1}; & x < 0 \end{cases} \\ &= \operatorname{sgn}(x) n |x|^{n-1}\end{aligned}$$

where $\operatorname{sgn}(x)$ is the function defined as

$$\operatorname{sgn}(x) = \begin{cases} 1; & x > 0 \\ 0; & x = 0 \\ -1; & x < 0 \end{cases}$$

For the point $x = 0$, going back to the definition of the derivative, we have

$$\begin{aligned}\lim_{h \to 0} \frac{f(0+h) - f(0)}{h} &= \lim_{h \to 0} \frac{|h|^n}{h} \\ &= \lim_{h \to 0} \operatorname{sgn}(h)|h|^{n-1} \\ &= 0 \qquad (\text{since } \operatorname{sgn}(h) \text{ is either 1 or } -1 \text{ when } h \text{ approaches } 0)\end{aligned}$$

So the final answer may be written as $\frac{d}{dx}|x|^n = \operatorname{sgn}(x) n |x|^{n-1}$.

5. His change in weight with respect to altitude is $\frac{dw}{dR} = -\frac{4 \times 10^9}{R^3}$. His change in altitude with respect to time is $\frac{dR}{dt} = 16$ miles /week. By the chain rule, his change in weight with respect to time is

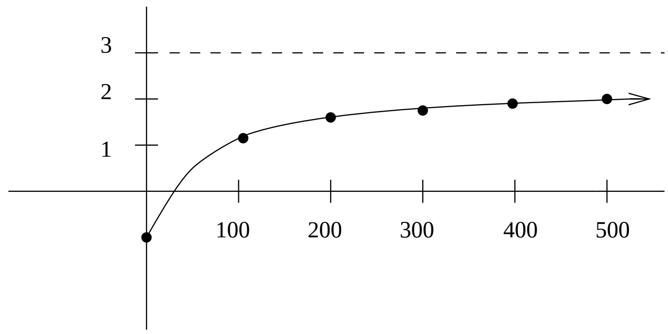
$$\frac{dw}{dt} = \frac{dw}{dR}\frac{dR}{dt} = -\frac{4 \times 10^9}{R^3} \times 16 = -\frac{64 \times 10^9}{R^3}$$

His change in weight due to climbing will therefore be $-\frac{64\times 10^9}{R^3} - 4$ so we wish to solve

$$-5 = \frac{-64\times 10^9}{R^3} - 4 \Rightarrow R^3 = 64\times 10^9 \Rightarrow R = 4\times 10^3 \text{ miles}$$

At $R = 4\times 10^3$ miles from the earth's centre, his weight loss will be the same by either method. From the equation for $\frac{dw}{dR}$ we can see that at a higher altitude he would lose more weight by heavy exercise.

6. The following is a sketch of $P(x)$:



The slope of the line joining $P(x)$ to the origin is highest when it is a tangent line for the function, so we need to compute which tangent line of $P(x)$ goes through the origin.

Given an arbitrary point $(x_0, P(x_0))$, the tangent line has equation

$$y - \left(\frac{3x_0 - 200}{x_0 + 400}\right) = (x - x_0)\frac{1400}{(x_0+400)^2}$$

since $P'(x) = \frac{1400}{(x+400)^2}$. If the tangent line goes through $(0, 0)$, we must have

$$-\frac{(3x_0 - 200)}{x_0 + 400} = -\frac{1400x_0}{(x_0+400)^2}$$

Solving for x_0,

$$(3x_0 - 200)(x_0 + 400) = 1400x_0$$
$$3x_0^2 - 400x_0 - 80000 = 0$$

Using the quadratic formula, the solutions are

$$x_0 = \frac{400 \pm \sqrt{160000 + 960000}}{6}$$
$$x_0 \approx 243, \ -110$$

Ignoring the negative solution (which doesn't make sense) we see that the average profit is maximized when $x \approx 243$, about 243 kg are sold.

7. Suppose n is a negative integer.

$$\frac{d}{dx}(x^n) = \frac{d}{dx}\left(\frac{1}{x^{-n}}\right) = \frac{(0)(x^{-n}) - (1)(-n)(x^{-n-1})}{x^{-2n}}$$

by the quotient rule and the formula above since $-n$ is a positive integer.

$$= \frac{nx^{-n-1}}{x^{-2n}}$$
$$= nx^{-n-1+2n}$$
$$= nx^{n-1}$$

8. (a) Differentiating both sides of the formula gives

$$1 + 2x + 3x^2 + \cdots + nx^{n-1} = \frac{nx^{n+1} - (n+1)x^n + 1}{(x-1)^2}$$

(b) Multiplying both sides of the above equation by x, we have

$$x + 2x^2 + 3x^3 + \cdots + nx^n = \frac{nx^{n+2} - (n+1)x^{n+1} + x}{(x-1)^2}$$

Differentiating gives
$1^2 + 2^2x + 3^2x^2 + \cdots + n^2x^{n-1}$ = the horrible derivative of the RHS.

9. Computing derivatives, we have

$$n = 1: \quad \frac{dy}{dx} = \frac{(x+1)(1) - x(1)}{(x+1)^2} = \frac{1}{(x+1)^2}$$
$$n = 2: \quad \frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{1}{(x+1)^2}\right) = \frac{(x+1)^2(0) - 2(x+1)}{(x+1)^4} = \frac{-2}{(x+1)^3}$$
$$n = 3: \quad \frac{d^3y}{dx^3} = \frac{(x+1)^3(0) + (2)(3)(x+1)^2}{(x+1)^6} = \frac{2\cdot 3}{(x+1)^4}$$
$$n = 4: \quad \frac{d^4y}{dx^4} = \frac{(x+1)^4(0) + (2)(3)(4)(x+1)^3}{(x+1)^8} = -\frac{2\cdot 3\cdot 4}{(x+1)^5}$$

It is clear that the pattern emerging is

$$\frac{d^ny}{dx^n} = \frac{(-1)^{n+1}n!}{(x+1)^{n+1}}$$

where $n! = n(n-1)(n-2)\ldots(2)(1)$.

10. (a)

$$\frac{dm}{Pdv} = m_0\left(-\frac{1}{2\left(1-\frac{v^2}{c^2}\right)^{\frac{3}{2}}}\right)\left(-\frac{2v}{c^2}\right)$$
$$= \frac{m_0 v}{c^2\left(1-\frac{v^2}{c^2}\right)^{\frac{3}{2}}}$$

(b) The mass of the object will be increasing at a faster rate the closer the velocity is to the speed of light.

11. The information tells us that at the present moment, $\frac{dc}{dq} = 12$ and

$$\frac{dq}{dt} = \frac{\text{change in units}}{\text{change in time (in hours)}} = 2.$$

So, at the present moment, the chain rule gives $\frac{dc}{dt} = \frac{dc}{dq}\frac{dq}{dt} = (12)(2) = 24$. $\frac{dc}{dt}$ is the rate of change of cost with respect to time, so, at the present moment the cost is increasing by \$24 per hour.

12. Using the chain rule, differentiating with respect to time,

$$\frac{dF}{dt} = \frac{dF}{dr}\frac{dr}{dt}$$

Differentiating, we have $\frac{dF}{dr} = -\frac{2GmM}{r^3}$ and we know $\frac{dr}{dt} = -100km/h$ (negative since r is decreasing), so

$$\frac{dF}{dt} = -100\left(-\frac{2GmM}{r^3}\right)$$
$$= -\frac{100(-2)(6.67\times 10^{-20})(5)(5.98\times 10^{24})}{(6375)^3}$$
$$= 0.0015 \text{ Newtons per hour.}$$

13. The derivative of $\underbrace{x^2+x^2+\cdots+x^2}_{x \text{ times}}$ is *not* $\underbrace{2x+2x+\cdots+2x}_{x \text{ times}}$ since the number of terms in the sum $2x+2x+\cdots+2x$ is increasing if x is increasing. In fact, if we differentiate $x\cdot x^2$ using the product rule, we have $x^2 + x2x = x^2 + 2x^2 = 3x^2$.