Supplementary Questions for HP Chapter 11

1. find the two straight lines through the point $(1, 0)$ that are tangent lines to the curve $y = x^3$.

2. The normal line to a curve at a point on the curve is the straight line which passes through that point and is perpendicular to the tangent line at that point.

(a) Find the equation of the normal line to the curve $y = x^2$ at the point $(2, 4)$.

(b) Find the equation of the normal line to the curve $y = x^2$ at an arbitrary point (x_0, x_0^2).

(c) At what point(s) on the curve $y = x^2$ does the normal line pass through the point $(2, \frac{1}{2})$ in the x-y plane?

3. (a) Show that the curve $y = x^5 + 2x$ has no horizontal tangent line.

(b) Find constants a, b, so that the curve $y = x^2 + ax + b$ has a horizontal tangent line at the point $(0, 1)$.

4. Find a formula for $\frac{d}{dx}|x|^n$ where $n > 1$ is an integer.

5. John's weight in pounds is given by the formula $w = 2 \times 10^9 / R^2$ where R is his distance, in miles, from the center of the earth.

Suppose John can lose an average of 5 pounds a week by heavy exercise, and an average of 4 pounds a week by climbing mountains, where he can climb at a rate of 16 miles of altitude per week.

At what radius from the center of the earth will his weight loss by climbing (*including* his reduction in weight due to increased altitude) be the same as his weight loss by exercising? By which method could he lose more weight if he climbs higher than the radius found above?

6. A manufacturer's profit from the sale of x kilograms of a commodity per week is given by

$$P(x) = \frac{3x - 200}{x + 400}$$

The average profit per kilogram when x kilograms are sold is given by

$$p(x) = \frac{P(x)}{x}$$

If a point $(x, P(x))$ on the total profit curve is joined to the origin by a straight line, the slope of this line is the average profit $p(x)$ for that x. Use this idea to find the sales level for highest average profit. (A sketch of $P(x)$ may help.)

7. On page 615 of the text, it is shown that $\frac{d}{dx}(x^n) = nx^{n-1}$ where n is a positive integer. Use the quotient rule to show that this is true for negative integers n.

8. Given the formula

$$1 + x + x^2 + \cdots + x^n = \frac{x^{n+1} - 1}{x - 1} \qquad (x \neq 1)$$

determine by differentiation a formula for

(a) $1 + 2x + 3x^2 + \cdots + nx^{n-1}$

(b) $1^2 + 2^2x + 3^2x^2 + \cdots + n^2x^{n-1}$

9. Find a formula for the n^{th} derivative of $y = \frac{x}{(x+1)}$.

10. The Theory of Relativity predicts that an object whose mass is m_0 when it is at rest will be heavier when moving near the speed of light. When the object is moving at velocity v, its mass m is given by

$$m = \frac{m_0}{\sqrt{1 - \frac{v^2}{c^2}}}$$

where c is the speed of light.

(a) Find $\frac{dm}{dv}$

(b) In terms of physics, what does $\frac{dm}{dv}$ tell you?

11. A firm computes that at the present moment its output q is increasing at a rate of two units per hour and that its marginal cost $\frac{dc}{dq}$ is 12. At what rate is its cost increasing per hour? Explain your answer.

12. When a mass of 5 kg is r meters from the centre of the earth, the magnitude of the force of attraction of the earth on m is

$$F = G\frac{m}{r^2}M$$

where $M = 5.98 \times 10^{24}$ kg is the mass of the earth, and $G = 6.67 \times 10^{-20} \frac{\text{Newtons}\cdot\text{km}^2}{\text{kg}^2}$. If m is falling at 100 km /h when it is 5 km above the surface of the earth, how fast is F changing? (The radius of the earth is 6370 km.)

13. What is wrong with the following proof that $3 = 2$?

$$x^3 = x \cdot x^2$$
$$x^3 = \underbrace{x^2 + x^2 + \cdots + x^2}_{x \text{ times}}$$

Differentiating,

$$\begin{aligned}
3x^2 &= \underbrace{2x + 2x + \cdots + 2x}_{x \text{ times}} \\
3x^2 &= x2x \\
3x^2 &= 2x^2 \\
3 &= 2
\end{aligned}$$