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Soln

Department of Mathematics
University of Toronto

Tuesday, Mar. 6, 2018, 6:10-8:00 PM
MAT 133Y TERM TEST #3
Calculus and Linear Algebra for Commerce
Duration: 1 hour 50 minutes

Aids Allowed: A TI-30X IIS calculator, to be supplied by student. **No other calculator is permitted.**

Instructions: Fill in the information on this page, and make sure your test booklet contains 10 pages. In addition, you should have a **multiple-choice answer sheet**, on which you should fill in your name, number, tutorial time, tutorial room, and tutor's name.

This test consists of 10 multiple choice questions, and 4 written-answer questions.
For the **multiple choice questions** you can do your rough work in the test booklet, but you must record your answer by circling the appropriate letter **on the answer sheet** with your pencil. Each correct answer is worth 4 marks; a question left blank, or an incorrect answer, or two answers for the same question is worth 0. For the **written-answer questions**, present your solutions in the space provided. The value of each written-answer question is indicated beside it.

ENCLOSE YOUR FINAL ANSWER IN A BOX AND WRITE IT IN INK.

TOTAL MARKS: 100

FAMILY NAME: ______________________

GIVEN NAME: ______________________

STUDENT NO: ______________________

SIGNATURE: ______________________

TUTORIAL TIME and ROOM: ______________________

REGCODE and TIMECODE: ______________________

T.A.'S NAME: ______________________

Regcode	Timecode	Room	**Regcode**	Timecode	Room
T0101	M9A	RS310	T0502	W3B	GB120
T0102	M9B	BA2135	T0503	W3C	BA1240
T0103	M9C	HA316	T0601	R4A	BA2195
T0104	M9D	LM157	T0602	R4B	SS2106
T0201	M3A	UC52	T0603	R4C	BA1240
T0202	M3B	UC87	T0604	R4D	SS2105
T0203	M3C	BA3012	T0701	F2A	BA2135
T0204	M3D	SS2127	T0702	F2B	SS2105
T0301	T3A	MS3278	T0703	F2C	BA2165
T0302	T3B	UC144	T0801	F3A	BA1240
T0303	T3C	BA1220	T0802	F3B	SS2105
T0304	T3D	BA3012	T0803	F3C	RW143
T0401	W9A	AB107	T5101	M5A	MS4171
T0402	W9B	BA2185	T5102	M5B	BA2175
T0403	W9C	LM157	T5103	M5C	BA2139
T0501	W3A	BA3116			

FOR MARKER ONLY	
Multiple Choice	
B1	
B2	
B3	
B4	
TOTAL	

NAME: ______________________ STUDENT NO: ______________

PART A. Multiple Choice

1. *[4 marks]*

How many inflection points does the graph of $f(x) = x^4 - 6x^2 + 8$ have?

$f'(x) = 4x^3 - 12x$
$f''(x) = 12x^2 - 12 = 12(x^2-1)$
$= 12(x-1)(x+1)$ which changes sign at $x=1$ and $x=-1$.
2 p.o.i. A

A. 2

B. 4

C. 1

D. 3

E. 0

2. *[4 marks]*

How many vertical asymptotes does the function $f(x) = \ln|x^2 - 3x + 1| + \dfrac{1}{(x+1)e^x}$ have?

V.A. at $x=-1$ because $x+1=0$.
$\ln|x|$ has V.A. when $x=0$
so $\ln|x^2-3x+1|$ has a V.A.
when $x^2-3x+1=0$
$x = \dfrac{3 \pm \sqrt{9-4}}{2}$ i.e. at 2 values of x since $5>0$.
There are 3 V.A.s altogether
D

A. 0

B. 1

C. 2

D. 3

E. 4

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3. *[4 marks]*

If $f''(x) = 3x^2 + 2$, $f'(0) = 4$, and $f(0) = 2$, then $f(2) =$

A. 14

B. 8

C. 20

D. 18

E. 10

$f'(x) = x^3 + 2x + C$

$4 = f'(0) = C$ so $f'(x) = x^3 + 2x + 4$

$f(x) = \frac{x^4}{4} + x^2 + 4x + D$

$2 = f(0) = D$

So $f(x) = \frac{x^4}{4} + x^2 + 4x + 2$

$f(2) = \frac{16}{4} + 4 + 8 + 2 = 18$ D

4. *[4 marks]*

If $\int_1^5 f(x)dx = 3$, $\int_1^2 f(x)dx = -6$, and $\int_3^5 f(x)dx = 1$, then $\int_2^3 f(x)dx =$

A. -8

B. -4

C. -2

D. 8

E. -10

$\int_1^5 f(x)dx = \int_1^2 f(x)dx + \int_2^3 f(x)dx + \int_3^5 f(x)dx$

$3 = -6 + \int_2^3 f(x)dx + 1$

$\int_2^3 f(x)dx = 8$ D

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5. *[4 marks]*

If $f(x) = \int_1^x (\ln|t|)^4\, dt$, then $f'(-e) =$

By the fundamental theorem

$f'(x) = (\ln|x|)^4$

$f'(e) = (\ln|-e|)^4 = (\ln e)^4 = 1^4$

$= 1$ E

A. -4

B. 4

C. e^4

D. 0

E. 1

6. *[4 marks]*

The average value of $f(x) = (x-1)(3-x)$ on the interval $[1,3]$ is

$\bar{f} = \frac{1}{3-1}\int_1^3 (x-1)(3-x)\,dx$

One could expand and integrate

or $= \frac{1}{2}\int_0^2 u(2-u)\,du$

Let $u = x-1$
$du = dx$
$-x = -u-1$
$3-x = 2-u$

$= \frac{1}{2}\int_0^2 (2u-u^2)\,du$

$= \frac{1}{2}\left[u^2 - \frac{u^3}{3}\right]_0^2$

$= \frac{1}{2}\left[4-\frac{8}{3}\right]$

$= \frac{2}{3}$ C

A. 2

B. $\frac{4}{3}$

C. $\frac{2}{3}$

D. 1

E. $\frac{1}{2}$

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7. *[4 marks]*

$\int_1^4 \left(\frac{1}{\sqrt{x}} + \sqrt{x}\right) dx =$ $\left[2x^{\frac{1}{2}} + \frac{2}{3}x^{3/2}\right]_1^4$

$= \left(2 \cdot 2 + \frac{2}{3} \times 8\right) - \left(2 + \frac{2}{3}\right)$

$= 2 + \frac{2}{3}(8-1) = 2 + \frac{14}{3} = \frac{20}{3}$ A

A. $\frac{20}{3}$

B. $\frac{11}{3}$

C. 4

D. 3

E. $\frac{1}{2}$

8. *[4 marks]*

$\int_{-2}^{-1} \frac{dx}{2x+1}$ $= \frac{1}{2} \ln|2x+1| \Big|_{-2}^{-1}$

A. does not exist because $\ln x$ is not defined for $x < 0$

B. is $-\ln 3$

C. is $-\ln\sqrt{3}$

D. is $-\frac{2}{3}$

E. is $-\frac{1}{3}$

$= \frac{1}{2}\left[\ln(|-1|) - \ln(|-3|)\right]$

$= \frac{1}{2}\left[0 - \ln 3\right]$

$= -\frac{1}{2}\ln 3$

$= -\ln\sqrt{3}$ C

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9. *[4 marks]*

If 80 people are put on an island and in 5 years there are 120 people, then how many people will there be in 12 years assuming the rate of growth of the population is proportional to the population?

$\frac{dN}{dt} = kN \quad \int \frac{dN}{N} = \int k\,dt \quad \ln N = kt + C$

$N = Ae^{kt}$

$N(0) = 80$ so $N(t) = 80e^{kt}$

$N(5) = 120 = 80e^{5k}$

$\left(\frac{3}{2}\right) = e^{5k} \quad \left(\frac{3}{2}\right)^{\frac{1}{5}} = e^{k}$

$N(t) = 80\left(\frac{3}{2}\right)^{\frac{t}{5}}$

$N(12) = 80\left(\frac{3}{2}\right)^{\frac{12}{5}} \approx 211.69$

D

A. 96

B. 35, 831, 808

C. 16, 201, 437

D. 212

E. 1004

10. *[4 marks]*

For $q \geq 0$, the demand function for a product is given by $q = \frac{90}{p} - 2$ and the supply function is given by $q = p - 1$. At market equilibrium, producer surplus is closest to

A. 32

B. 28

C. 73

D. 56

E. 45

Equilibrium is at

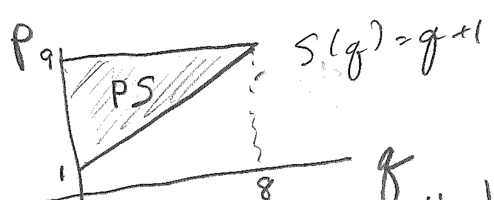
$p - 1 = \frac{90}{p} - 2$

$p(p+1) = 90$

$p^2 + p - 90 = 0 \quad (p+10)(p-9) = 0$

So $p_0 = 9$ and $q_0 = 8$

$p = q + 1$



P.S is the area of a triangle with height 8 and base 8.

$P.S. = \frac{1}{2} 8 \times 8 = 32$ A

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PART B. Written-Answer Questions

1. *[16 marks]*

Sketch the graph of $y = x^4e^x$
given that $y' = x^3e^x(x+4)$
and $y'' = x^2e^x(x+2)(x+6)$

Make sure to indicate all vertical and/or horizontal asymptotes, where y is increasing or decreasing, local extrema, where y is concave up or down, and all points of inflection.

The fcn. is cont. everywhere, so $\boxed{\text{no V.A.}}$

$\lim_{x\to\infty} x^4e^x = \infty$

$\lim_{x\to-\infty} x^4e^x = \lim_{x\to-\infty} \frac{x^4}{e^{-x}} = \frac{\infty}{\infty}$ so L'Hôp 4 times

$= \lim_{x\to-\infty} \frac{24}{e^{-x}} = 0$

$\boxed{y=0 \text{ is H.A. at } -\infty}$

Crit pts at $x=0$ and $x=-4$

	x	y'	
$(-\infty,-4)$	-5	+	↑
$(-4,0)$	-1	−	↓
$(0,\infty)$	1	+	↑

$\boxed{x=-4 \text{ is a local max}}$

$\boxed{x=0 \text{ is a local min}}$

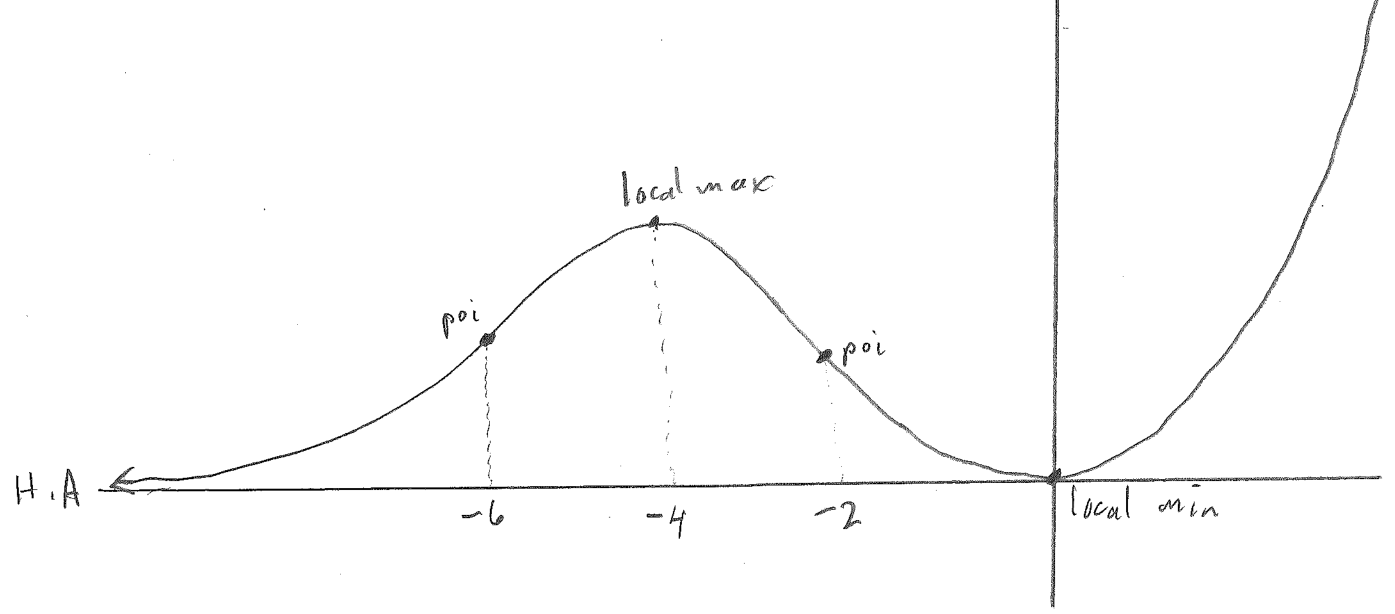
in fact global min

$y''=0$ at $x=0$, $x=-2$, $x=-6$

	x	y''	y
$(-\infty,-6)$	-7	+	conc up
$(-6,-2)$	-3	−	conc down
$(-2,0)$	-1	+	conc up
$(0,\infty)$	1	+	conc up

$\boxed{x=-6 \text{ and } x=-2 \text{ are poi.}}$

$x=0$ isn't.



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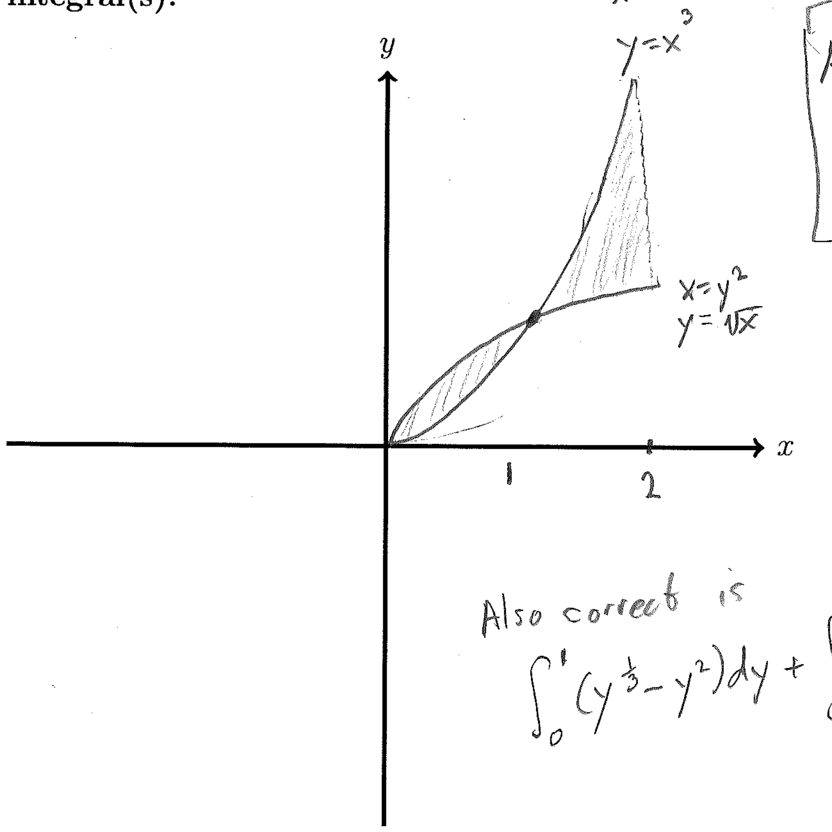
2. *[16 marks]*

[8] (a) Graph the curves: $x = y^2$ and $y = x^3$ on the axes below showing intersection points clearly. Without using absolute values, write the area of the region(s) between the curves from $x = 0$ to $x = 2$ using definite integral(s) but **do not evaluate the integral(s).**



$$\boxed{A = \int_0^1 (\sqrt{x} - x^3)\,dx + \int_1^2 (x^3 - \sqrt{x})\,dx}$$

Also correct is

$$\int_0^1 (y^{\frac{1}{3}} - y^2)\,dy + \int_1^2 (x^3 - \sqrt{x})\,dx$$

[8] (b) Find the accumulated value at the end of 10 years of a continuous annuity with payments at time t at the rate of $f(t) = 100t$ per year and interest at 4% compounded continuously.

$$\text{F.V.} = \int_0^{10} 100 t e^{.04(10-t)}\,dt = 100 e^{.4} \int_0^{10} t e^{-.04t}\,dt$$

Let $u = t \quad dv = e^{-.04t}$

$du = dt \quad v = -\frac{e^{-.04t}}{.04}$

$$\text{F.V.} = 100 e^{.4}\left[-\frac{t e^{-.04t}}{.04} \Big|_0^{10} + \frac{1}{.04}\int_0^{10} e^{-.04t}\,dt \right]$$

$$= 100 e^{.4}\left[-\frac{10}{.04} e^{-.4} - \frac{1}{(.04)^2} e^{-.04t} \Big|_0^{10} \right]$$

$$= 100 e^{.4}\left[-\frac{10}{.04} e^{-.4} - \frac{1}{(.04)^2} e^{-.4} + \frac{1}{(.04)^2} \right]$$

$$= 100\left[-\frac{10}{.04} - \frac{1}{(.04)^2} + \frac{e^{.4}}{(.04)^2} \right]$$

$$= \boxed{5,739.04}$$

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3. *[16 marks]* Evaluate the following integrals.

[5] (a) $\int_0^1 e^x\sqrt{e^x+2}\,dx = \int_3^{e+2} \sqrt{u}\,du$

Let $u = e^x+2$

$du = e^x dx$

$= \frac{2}{3}u^{3/2}\Big|_3^{e+2}$

$= \boxed{\frac{2}{3}\left[(e+2)^{3/2} - 3^{3/2}\right]}$

[5] (b) $\int_0^1 \frac{x^2}{x+1}\,dx = \int_0^1 \left(x-1+\frac{1}{x+1}\right)dx$

$$\begin{array}{r} x-1 \\ x+1\,\overline{)\,x^2 \qquad} \\ \underline{x^2+x} \\ -x \\ \underline{-x-1} \\ 1 \end{array}$$

$= \left[\frac{x^2}{2} - x + \ln|x+1|\right]_0^1$

$= \frac{1}{2} - 1 + \ln 2 = \boxed{-\frac{1}{2} + \ln 2}$

[6] (c) $\int_2^3 \frac{x+1}{x^2-x}\,dx =$

$\frac{x+1}{x(x-1)} = \frac{A}{x} + \frac{B}{x-1}$

$A(x-1) + Bx = x+1$

$x=1 \quad B = 2$

$x=0 \quad -A = 1 \quad A = -1$

$\int_2^3 \left(-\frac{1}{x} + \frac{2}{x+1}\right)dx$

$= \left(-\ln|x| + 2\ln|x-1|\right)_2^3$

$= \boxed{-\ln\left(\frac{3}{2}\right) + 2\ln 2}$

$= \ln\left(\frac{2}{3}\right) + \ln 4$

$= \ln\left(\frac{8}{3}\right)$

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4. *[12 marks]* A factory must determine how many widgets to produce each month. Assuming that monthly production is at least 100 widgets and no more than 2000 widgets, the price per widget (in thousands of dollars) at which q widgets can be sold is given by

$$p = \frac{2}{\sqrt{q}} + \frac{100}{q}.$$

The cost of producing q widgets is given by

$$c(q) = \begin{cases} 9q^{1/3} & 100 \leq q \leq 1000 \\ 90 & 1000 \leq q \leq 2000 \end{cases}.$$

How many widgets should the factory produce to maximize profit? Remember to justify your answer.

$$\pi(q) = pq - C$$
$$= 2\sqrt{q} + 100 - \begin{cases} 9q^{\frac{1}{3}} & 100 \leq q \leq 1000 \\ 90 & 1000 \leq q \leq 2000 \end{cases}$$

Since $9q^{\frac{1}{3}} = 90$ at $q = 1000$, $\pi(q)$ is cont. on $[100, 2000]$, hence must have a max on the interval, either at $q = \boxed{100}$, $q = \boxed{2000}$, or at a crit pt in $(100, 2000)$

$$\frac{d\pi}{dq} = \begin{cases} \frac{1}{\sqrt{q}} - \frac{3}{q^{2/3}} & 100 < q < 1000 \\ \frac{1}{\sqrt{q}} & 1000 < q < 2000 \end{cases}$$

and doesn't exist at $q = 1000$.

crit pts are $\boxed{q = 1000}$

and $\frac{1}{\sqrt{q}} - \frac{3}{q^{2/3}} = 0$

$q^{2/3} - 3q^{\frac{1}{2}} = 0$

$q^{2/3} = 3q^{\frac{1}{2}}$ put both sides to 6th power

$q^4 = 3^6 q^3$

$q = 3^6 = \boxed{729}$

$\pi(100) = 78.33$

$\pi(1000) = 73.25$

max $\boxed{\pi(2000) = 99.44}$

$\pi(729) = 73$

You could also notice that π actually has a local min at $q = 729$ so it isn't necessary to test it there.

$\boxed{\text{max at } X = 2000}$