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Department of Mathematics
University of Toronto
Tuesday, March 7, 2017, 6:10-8:00 PM
MAT 133Y TERM TEST #3
Calculus and Linear Algebra for Commerce
Duration: 1 hour 50 minutes

Soln.

Aids Allowed: A TI-30X IIS calculator, to be supplied by student. **No other calculator is permitted.**

Instructions: Fill in the information on this page, and make sure your test booklet contains 10 pages. In addition, you should have a **multiple-choice answer sheet**, on which you should fill in your name, number, tutorial time, tutorial room, and tutor's name.

This test consists of 10 multiple choice questions, and 4 written-answer questions.
For the **multiple choice questions** you can do your rough work in the test booklet, but you must record your answer by circling the appropriate letter **on the answer sheet** with your pencil. Each correct answer is worth 4 marks; a question left blank, or an incorrect answer, or two answers for the same question is worth 0. For the **written-answer questions**, present your solutions in the space provided. The value of each written-answer question is indicated beside it.

ENCLOSE YOUR FINAL ANSWER IN A BOX AND WRITE IT IN INK.

TOTAL MARKS: 100

FAMILY NAME: ____________________

GIVEN NAME: ____________________

STUDENT NO: ____________________

SIGNATURE: ____________________

TUTORIAL TIME and ROOM: ____________________

REGCODE and TIMECODE: ____________________

T.A.'S NAME: ____________________

Regcode	Timecode	Room	**Regcode**	Timecode	Room
T0101	M9A	HA316	T0502	W3B	PB255
T0102	M9B	HA401	T0503	W3C	UC87
T0103	M9C	HA410	T0601	R4A	BA3012
T0104	M9D	LM157	T0602	R4B	AP120
T0201	M3A	MS4171	T0603	R4C	BA B024
T0202	M3B	UC87	T0604	R4D	BF215
T0203	M3C	GB303	T0701	F2A	BA2145
T0204	M3D	WB119	T0702	F2B	BA2155
T0301	T3A	HA401	T0703	F2C	BA2165
T0302	T3B	UC52	T0801	F3A	BA2145
T0303	T3C	ES B142	T0802	F3B	BA2155
T0304	T3D	GB303	T0803	F3C	BA2165
T0401	W9A	AB114	T5101	M5A	AB114
T0402	W9B	BA2159	T5102	M5B	AP120
T0403	W9C	BA2175	T5103	M5C	BA3116
T0501	W3A	GB248	T5104	M5D	HA316

FOR MARKER ONLY	
Multiple Choice	
B1	
B2	
B3	
B4	
TOTAL	

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PART A. Multiple Choice

1. *[4 marks]*

The difference of 2 numbers is 12. The product of the 2 numbers has

A. an absolute minimum value of 6.

B. an absolute minimum value of 36.

C. an absolute minimum value of -6.

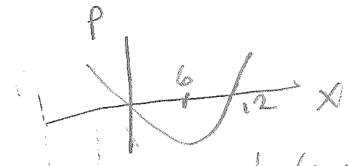
D. an absolute minimum value of -36.

E. no absolute minimum value.

$x-y=12 \quad x-12=y$

$P=xy=x(x-12)$

$\frac{dP}{dx}=2x-12=0$ at $x=6$



P has an absolute min at $x=6$, $y=-6$

$P=-36$

(D)

2. *[4 marks]*

If $f(x)=x^3+ax^2+bx+c$, where a, b and b are constants, has a point of inflection at $x=0$, then

A. $a=0$.

B. $a>0$.

C. $a<0$.

D. a could be any real number.

E. $b=0$ but a can be any real number.

$f'(x)=3x^2+2ax+b$

$f''(x)=6x+2a$

$f''(0)=2a$

For f to have a pt. of inflection at $x=0$, $f''(0)=0$, so $\boxed{a=0}$

If $a=0$, $f''(x)=6x$ which does indeed change signs at $x=0$.

(A)

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3. *[4 marks]*

If $f(x) = \ln[g(x)]$, $g(x) > 0$ and $g(x)$ is always increasing and concave downward on some interval, then on that interval $f(x)$ is

[✓ **A.**] always increasing and concave downward.

B. always decreasing and concave downward.

C. always increasing and concave upward.

D. always decreasing and concave upward.

E. always increasing but sometimes concave upward and sometimes concave downward.

$f''(x) = \frac{g'(x)}{g(x)} > 0$ because g and $g' > 0$ so $\boxed{f \text{ increasing}}$

$f''(x) = \frac{g(x)g''(x) - [g'(x)]^2}{[g(x)]^2} < 0$ because $gg'' < 0$ (g is concave down)

and $-[\ ]^2 < 0$

so $gg'' - [g']^2 < 0$ and $[g(x)]^2 > 0$.

$f''(x) < 0$ always so $\boxed{f \text{ is concave down}}$ Ⓐ

4. *[4 marks]*

The graph of $f(x) = \dfrac{\sqrt{4x^2+3}}{x}$ has

A. horizontal asymptote $y = 1$ only.

B. horizontal asymptote $y = 2$ only.

C. horizontal asymptote $y = -2$ only.

D. horizontal asymptotes $y = -2$ and $y = 2$.

E. no horizontal asymptotes.

$$\lim_{x\to\infty} \frac{\sqrt{4x^2+3}}{x} = \lim_{x\to\infty} \sqrt{\frac{4x^2+3}{x^2}}$$

$$= \lim_{x\to\infty} \sqrt{4+\frac{3}{x^2}} = 2$$

so $\boxed{\text{H.A at } +\infty \text{ is } y=2}$

$$\lim_{x\to-\infty} \frac{\sqrt{4x^2+3}}{x}$$

$$= \lim_{x\to-\infty} \frac{\sqrt{4x^2+3}}{-\sqrt{x^2}} \quad \text{since } x<0$$

$$= \lim_{x\to-\infty} -\sqrt{4+\frac{3}{x^2}} = -2$$

so, $\boxed{\text{H.A. at } -\infty \text{ is } y=-2}$

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5. *[4 marks]*

If $F(x) = x \int_3^x e^{-t^2}\,dt$ then $F'(3) =$

A. xe^{-9}

B. $3x$

C. $9e^{-9}$

D. e^{-9}

E. $3e^{-9}$

$F'(x) = (x)' \int_3^x e^{-t^2} dt + x \frac{d}{dx} \int_3^x e^{-t^2} dt$ (prod rule)

$= \int_3^x e^{-t^2} dt + xe^{-x^2}$ Fund. Thm.

$F'(3) = \int_3^3 e^{-t^2} dt + 3e^{-9}$

$= 0 + 3e^{-9}$ $\boxed{E}$

6. *[4 marks]*

$$\int_0^2 \frac{x\,dx}{\sqrt{3x^2+4}} =$$

A. $\frac{1}{3}$

B. $\frac{2}{3}$

C. $\frac{1}{2}$

D. $\frac{3}{2}$

E. 4

$u = 3x^2+4$

$du = 6x\,dx$

$\frac{du}{6} = x\,dx$

$x=0 \Rightarrow u=4$

$x=2 \Rightarrow u=16$

$\frac{1}{6}\int_4^{16} \frac{1}{\sqrt{u}}\,du$

$= \frac{1}{6}\, 2 \cdot u^{\frac{1}{2}} \Big|_4^{16}$

$= \frac{1}{3}(4-2)$

$= \frac{2}{3}$ $\boxed{B}$

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7. *[4 marks]*

If $f'(x) = e^x$ and $f(0) = e^2$ then $f(2) =$

A. $2e^2 - 1$

B. $4e^2 + 1$

C. e^2

D. $e^2 - 1$

E. $2e^2$

$$f(x) = \int e^x dx = e^x + C$$

$$e^2 = f(0) = 1 + C \Rightarrow C = e^2 - 1$$

$$f(x) = e^x + e^2 - 1$$

$$f(2) = e^2 + e^2 - 1 = 2e^2 - 1$$

Ⓐ

8. *[4 marks]*

A continuous cashflow of \$525 per year at an annual interest rate of 2% compounded continuously for 20 years has a future value closest to

A. \$140,695

B. \$8,654

C. \$12,910

D. \$33,542

E. \$7,980

$$P.V. = \int_0^{20} 525 e^{-.02t} dt$$

$$F.V. = e^{.02 \times 20} \int_0^{20} 525 e^{-.02t} dt$$

$$= 525 e^{.40} \frac{e^{-.02t} \Big|_0^{20}}{-.02}$$

$$= \frac{525 e^{.40}}{.02} (1 - e^{-.40})$$

$$= \frac{525}{.02} (e^{.40} - 1)$$

$$\approx 12,910.40$$ Ⓒ

9. *[4 marks]*

The average value of $f(x) = x + \dfrac{1}{x}$ on the interval $[1, 3]$ is closest to

A. 1.54

B. 2.55

C. 2.67

D. 3.09

E. 2

$$\bar{f} = \frac{1}{3-1}\int_1^3 \left(x+\frac{1}{x}\right)dx$$
$$= \frac{1}{2}\left[\frac{x^2}{2} + \ln|x|\right]_1^3$$
$$= \frac{1}{2}\left[\left(\frac{9}{2}+\ln 3\right) - \left(\frac{1}{2}+\ln 1\right)\right]$$
$$= \frac{1}{2}\left[4+\ln 3\right] \approx 2.549$$

Ⓑ

10. *[4 marks]*

If $\dfrac{dy}{dx} = 2xy^2$ and $y(1) = 1$, then y is equal to

A. $\dfrac{1}{2 - x^2}$

B. $-\dfrac{1}{x^2}$

C. $\dfrac{2}{x^2 - 3}$

D. $e^{(x^2-1)}$

E. $x^2 - 1$

$$\frac{dy}{y^2} = 2x\,dx$$
$$-\frac{1}{y} = x^2 + C$$
$$-\frac{1}{1} = 1 + C$$
$$-2 = C$$
$$-\frac{1}{y} = x^2 - 2$$
$$\frac{1}{y} = 2 - x^2$$
$$y = \frac{1}{2-x^2}$$

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PART B. Written-Answer Questions

1. *[16 marks]*

If $f(x) = \dfrac{2x^2}{x^2-1}$, $f'(x) = \dfrac{-4x}{(x^2-1)^2}$, $f''(x) = \dfrac{12x^2+4}{(x^2-1)^3}$, find

[3] (a) any horizontal and/or vertical asymptote(s).

$\lim_{x\to\infty} f(x) = 2$ and the same at $-\infty$.

So $\boxed{y=2 \text{ is a H.A. at } +\infty \text{ and } -\infty}$

$\boxed{\text{V.A. at } x=1 \text{ and } x=-1}$

[3] (b) where f is increasing, decreasing, and all relative extrema,

The only crit. pt at which f is defined is $x=0$

When $x<0$ $f'>0$ where defined; and when $x>0$ $f'<0$ where defined.

$\boxed{\begin{array}{l} f \text{ increasing on } (-\infty,-1) \text{ and } (-1,0). \\ f \text{ decreasing on } (0,1),(1,\infty) \end{array}}$

	f'
$(-\infty,-1)$	+
$(-1,0)$	+
$(0,1)$	−
$(1,\infty)$	−

$\wedge$ $\boxed{\text{relative max at } x=0}$

[3] (c) where f is concave up, concave down, and all inflection points.

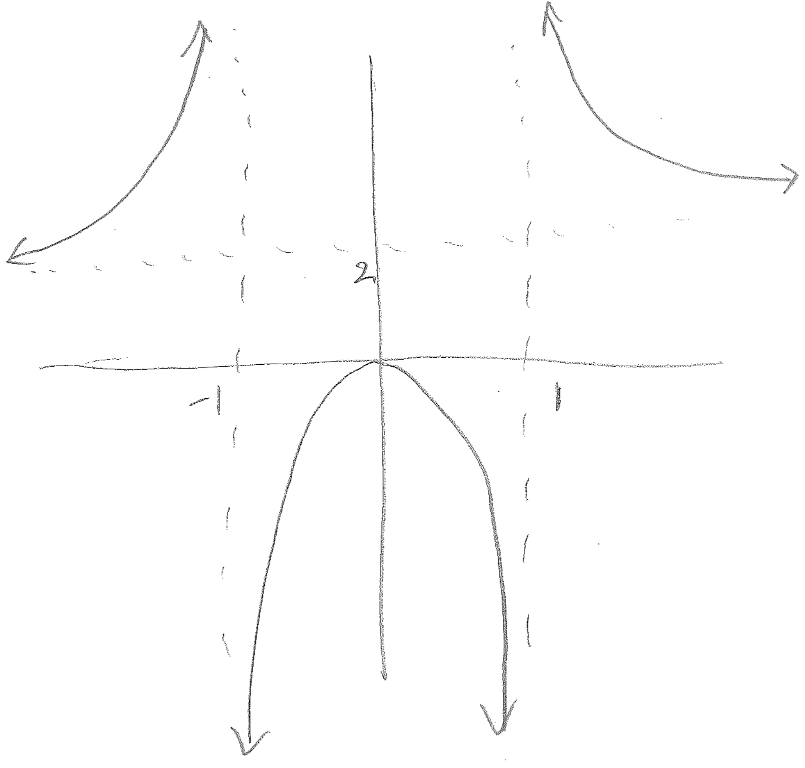
	f''	
$(-\infty,-1)$	+	conc up
$(-1,1)$	−	conc down
$(1,\infty)$	+	conc up

since $12x^2+4>0$ always

$x=\pm1$ are not pts on the curve

so not pts of inflection.

[7] (d) Sketch the curve, exhibiting the relevant information.



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2. *[14 marks]*

A manufacturer can produce **at most** 50 racing bicycles per quarter. The demand equation for the bicycles is

$$p = q^2 - 100q + 3200.$$

The average cost of producing the bicycles is

$$\bar{c} = \frac{2}{3}q^2 - 40q + \frac{10000}{q}.$$

How many bicycles should the manufacturer produce to maximize profit?
What is the absolute maximum profit?
Justify your answer by using one of the methods/tests we've learned.
(**Hint:** profit = total revenue - total cost)

$\bar{c} = \frac{c}{q}$ so $c = q\bar{c} = \frac{2}{3}q^3 - 40q^2 + 10{,}000$

and $r = pq = q^3 - 100q^2 + 3200q$

profit $\pi = r - c = \frac{1}{3}q^3 - 60q^2 + 3200q - 10{,}000$

$\frac{d\pi}{dq} = q^2 - 120q + 3200$
$= (q-40)(q-80)$
$= 0$ only at $q = 40$
since $0 \leq q \leq 50$

π is cont. on $[0, 50]$ so must have an absolute max at $q = 0, q = 50$
or the only crit pt $q = 40$

$\pi(0) = -10{,}000$
$\boxed{\pi(40) = 43{,}333}$ max
$\pi(50) = 41{,}667$

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3. *[14 marks]*

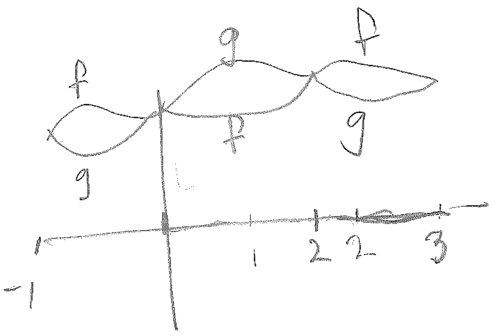
Find the area of the finite region bounded by $f(x) = 2x^2+3x+5$ and $g(x) = -x^2+9x+5$ from $x = -1$ to $x = 3$.

Intersect at

$2x^2+3x+5 = -x^2+9x+5$

$3x^2-6x = 0$

$3x(x-2)=0 \quad x=0$ and $x=2$



$$\text{Area} = \int_{-1}^{0}[f(x)-g(x)] + \int_0^2[g(x)-f(x)] + \int_2^3 f(x)-g(x)$$

because

$f(-\frac{1}{2}) = 1-\frac{3}{2}+5 = 4.5 \qquad g(-\frac{1}{2}) = -\frac{1}{4}-\frac{9}{2}+5 = .25$

$g(1) = 13$

$f(1) = 10$

$f(2.5) = 25 \qquad g(2.5) = 21.25$

$$\text{Area} = \int_{-1}^{0}(3x^2-6x)\,dx + \int_0^2(6x-3x^2)\,dx + \int_2^3(3x^2-6x)\,dx$$

$$= [x^3-3x^2]_{-1}^{0} + [3x^2-x^3]_0^2 + [x^3-3x^2]_2^3$$

$$= [0-(-4)] + [12-8] + [(27-27)-(8-12)]$$

$$= 4 + 4 + 4 = \boxed{12}$$

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4. *[16 marks]*

Evaluate the following integrals:

[8] (a) $\int_{-2}^{2} \frac{2-x}{3+x}dx$

Let $u = 3 + x$ $\quad du = 2x$

$-x = 3 - u$

$2 - x = 5 - u$

$= \int_1^5 \frac{5-u}{u}\,du$

$= \int_1^5 \left(\frac{5}{u} - 1\right)du = \left[5\ln|u| - u\right]_1^5 = (5\ln 5 - 5) - (-1)$

$= \boxed{5\ln 5 - 4}$

Alternatively $\quad x+3 \overset{-1}{\overline{)-x+2}}$

$\quad -x-3$

$\quad 5$

$\int = \int_{-2}^{2}\left(-1 + \frac{5}{x+3}\right)dx$

$= \left[-x + 5\ln|x+3|\right]_{-2}^{2} = (-2 + 5\ln 5) - (2 + 5\ln 1)$

$= \boxed{5\ln 5 - 4}$ (as before)

[8] (b) $\int_{4}^{\infty} \frac{1}{x^2-3x}dx \quad = \lim_{R\to\infty} \int_4^R \frac{dx}{x(x-3)}$

$\frac{1}{x(x-3)} = \frac{A}{x} + \frac{B}{x-3} \qquad A(x-3) + Bx = 1$

$x = 3 \Rightarrow B = \frac{1}{3}$

$x = 0 \Rightarrow A = -\frac{1}{3}$

$\lim_{R\to\infty} \int_4^R \frac{1}{3}\left[\frac{1}{x-3} - \frac{1}{x}\right]$

$= \lim_{R\to\infty} \frac{1}{3}\left[\ln|x-3| - \ln|x|\right]_4^R = \lim_{R\to\infty} \ln\left|\frac{x-3}{x}\right|_4^R$

$= \lim_{R\to\infty} \frac{1}{3}\ln\left|\frac{R-3}{R}\right| - \frac{1}{3}\ln\left(\frac{1}{4}\right)$ $\quad$ but $\frac{R-3}{R} \to 1$

so $\ln\left|\frac{R-3}{R}\right| \to 0$

$= -\frac{1}{3}\ln\left(\frac{1}{4}\right) = \boxed{\frac{1}{3}\ln 4}$