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Department of Mathematics
University of Toronto
Tuesday, March 8, 2016, 6:10-8:00 PM
MAT 133Y TERM TEST #3
Calculus and Linear Algebra for Commerce
Duration: 1 hour 50 minutes

Aids Allowed: A non-graphing calculator, with empty memory, to be supplied by student. No calculator may be used that has a button with $\frac{d}{dx}$ and/or $\int$ on it.

Instructions: Fill in the information on this page, and make sure your test booklet contains 10 pages. In addition, you should have a **multiple-choice answer sheet**, on which you should fill in your name, number, tutorial time, tutorial room, and tutor's name.

This test consists of 10 multiple choice questions, and 4 written-answer questions.
For the **multiple choice questions** you can do your rough work in the test booklet, but you must record your answer by circling the appropriate letter **on the answer sheet** with your pencil. Each correct answer is worth 4 marks; a question left blank, or an incorrect answer, or two answers for the same question is worth 0. For the **written-answer questions**, present your solutions in the space provided. The value of each written-answer question is indicated beside it.

ENCLOSE YOUR FINAL ANSWER IN A BOX AND WRITE IT IN INK.

TOTAL MARKS: 100

FAMILY NAME: ______________________

GIVEN NAME: ______________________

STUDENT NO: ______________________

SIGNATURE: ______________________

TUTORIAL TIME and ROOM: ______________________

REGCODE and TIMECODE: ______________________

T.A.'S NAME: ______________________

Regcode	Timecode	Room	Regcode	Timecode	Room
T0101A	M9A	BA2135	T0501B	W3B	SS2105
T0101B	M9B	BA2165	T0501C	W3C	UC52
T0101C	M9C	BA1240	T0601A	R4A	BL112
T0101D	M9D	BA2139	T0601B	R4B	BL114
T0201A	M3A	BA B024	T0601C	R4C	SS562
T0201B	M3B	RW142	T0601D	R4D	UC114
T0201C	M3C	WO25	T0701A	F2A	AP120
T0201D	M3D	WW119	T0701B	F2B	BF323
T0301A	T3A	ES4001	T0701C	F2C	LM155
T0301B	T3B	HA316	T0801A	F3A	AP120
T0301C	T3C	SS1086	T0801B	F3B	BF323
T0301D	T3D	SS2111	T0801C	F3C	LM155
T0401A	W9A	BA2195	T5101A	M5A	AP120
T0401B	W9B	AP120	T5101B	M5B	BA2175
T0401C	W9C	LM155	T5101C	M5C	BA2185
T0401D	W9D	BA2159	T5101D	M5D	SS2111
T0501A	W3A	HA316			

FOR MARKER ONLY	
Multiple Choice	
B1	
B2	
B3	
B4	
TOTAL	

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PART A. Multiple Choice

1. *[4 marks]*

The function $\dfrac{(x+1)(x-1)(x-3)}{(x-3)(x^2+1)}$ has:

A. horizontal asymptote(s) and three (3) vertical asymptotes.

B. horizontal asymptote(s) and one (1) vertical asymptote.

C. horizontal asymptote(s) and no vertical asymptotes.

D. no horizontal asymptotes and no vertical asymptotes.

E. no horizontal asymptotes and one (1) vertical asymptote.

The lead term in both numerator and denominator is x^3, so $y=1$ is a H.A. at both $+\infty$ and $-\infty$. $x=3$ is the only possible V.A., but since $x-3$ is in both num. and denom $\lim_{x \to 3} y = \frac{4 \cdot 2}{10}$ not a V.A. So Ⓒ.

2. *[4 marks]*

If $f''(x) = e^x + 2$ and $f'(0) = 1$ and $f(0) = 2$ then $f(2) =$

A. $e^2 + 2$

B. $e^2 + 3$

C. $e^2 + 4$

D. $e^2 + 5$

E. $e^2 + 7$

$f'(x) = e^x + 2x + C$

$1 = e^0 + C$ so $C = 0$

$f'(x) = e^x + 2x$

$f(x) = e^x + x^2 + D$

$2 = 1 + D$ so $D = 1$

$f(x) = e^x + x^2 + 1$

$f(2) = e^2 + 5$ Ⓓ

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3. *[4 marks]*

If $\int_1^6 g(x)dx = 4$ and $\int_2^6 g(x)dx = -5$, then $\int_1^2 g(x)dx =$

A. 1

B. 9

C. −1

D. −9

E. There is not enough information to find an answer.

$$\int_1^6 g(x)dx = \int_1^2 g(x)dx + \int_2^6 g(x)dx$$

$$4 = \int_1^2 g(x)dx + -5$$

$$9 = \int_1^2 g(x)dx$$ Ⓑ

4. *[4 marks]*

Suppose that $f'(8) = 6$ and $f'(0) = 1$. Then $\int_0^8 \frac{d^2 f}{dx^2} dx = \left.\frac{df}{dx}\right|_0^8 =$

A. 5

B. −5

C. 0

D. 6

E. C where C is an arbitrary constant.

$$= f'(8) - f'(0)$$

$$= 6 - 1 = 5$$ Ⓐ

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5. *[4 marks]*

If $F(x) = \int_4^x \frac{e^t}{\sqrt{t}} dt$ then $F'(1) =$

A. $e - \frac{e^2}{2}$

B. $\frac{e^2}{2} - e$

C. e

D. $-e$

E. $-\frac{e}{3}$.

$F'(x) = \frac{e^x}{\sqrt{x}}$

$F'(1) = \frac{e^1}{1} = e$

Ⓒ

6. *[4 marks]*

The area between $y = x^3$ and the y-axis from $y = -1$ to $y = 8$ is given by

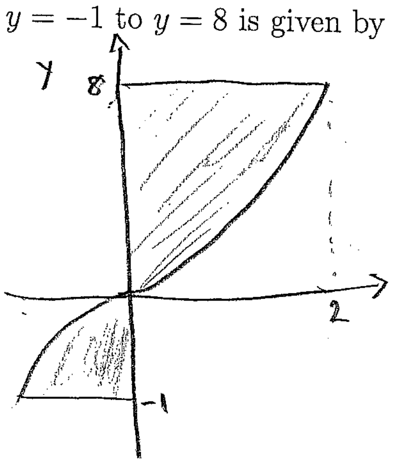
A. $\int_{-1}^{8} y^{\frac{1}{3}} dy$

B. $\int_{-1}^{2} x^3 dx$

C. $\int_0^8 (8 - x^3) dx + \int_{-1}^{0} (x^3 + 1) dx$

D. $\int_0^2 x^3 dx - \int_{-1}^{0} x^3 dx$

E. $\int_0^8 y^{\frac{1}{3}} dy - \int_{-1}^{0} y^{\frac{1}{3}} dy$



$y = x^3$

$x = y^{\frac{1}{3}}$

Integrating along y axis

$$\text{Area} = \int_{-1}^{0} (0 - y^{\frac{1}{3}}) dy + \int_0^8 (y^{\frac{1}{3}} - 0) dy$$

$$= -\int_{-1}^{0} y^{\frac{1}{3}} dy + \int_0^8 y^{\frac{1}{3}} dy$$ Ⓔ

Integrating along x-axis

$$\text{Area} = \int_{-1}^{0} (x^3+1) dx + \int_0^2 (8 - x^3) dx$$ Note: This is <u>not</u> C.

7. *[4 marks]*

The present value of a 15-year continuous annuity with payments at time t at the rate of $f(t) = \$2000$/year and interest at 3% compounded continuously is

A. \$167, 642.62

B. \$23, 875.87

C. \$37, 887.48

D. \$24, 158.12

E. \$37, 197.83

$$P.V. = \int_0^{15} 2000 e^{-.03t} dt$$

$$= \frac{2000}{-.03} e^{-.03t} \Big|_0^{15}$$

$$= \frac{2000}{-.03} \left[e^{-.45} - 1 \right]$$

$$= \frac{2000}{.03} \left[1 - e^{-.45} \right]$$

$$= \$24,158.12 \quad \textcircled{D}$$

8. *[4 marks]*

The average value of $f(x) = x\sqrt{x^2+16}$ on the interval $[0, 3]$ is

A. $\frac{61}{9}$

B. $\frac{125}{9}$

C. $\frac{61}{6}$

D. $\frac{125}{6}$

E. $\frac{61}{3}$

$$\text{Average} = \frac{1}{3-0} \int_0^3 x \sqrt{x^2+16}\, dx$$

Let $u = x^2+16$

$du = 2x dx \qquad \frac{du}{2} = x dx \qquad x=0 \quad u=16$

$x=3 \quad u=25$

$$Av = \frac{1}{3} \cdot \frac{1}{2} \int_{16}^{25} u^{\frac{1}{2}} du$$

$$= \frac{1}{6} \cdot \frac{2}{3} u^{3/2} \Big|_{16}^{25}$$

$$= \frac{1}{9} \left(25^{3/2} - 16^{3/2} \right)$$

$$= \frac{1}{9} (125 - 64) = \frac{61}{9} \quad \textcircled{A}$$

9. *[4 marks]*

$\int_0^2 xe^x dx =$

A. $e^2 + 1$

B. $e^2 - 1$

C. e^2

D. $1 - e^2$

E. $2e^2$

Let $u = x \quad dv = e^x dx$

$du = dx \quad v = e^x$

$$\int_0^2 xe^x dx = xe^x\Big|_0^2 - \int_0^2 e^x dx$$

$$= 2e^2 - e^x\Big|_0^2$$

$$= 2e^2 - (e^2 - 1)$$

$$= e^2 + 1$$

Ⓐ

10. *[4 marks]*

$\int_0^1 \frac{2-x}{2+x} dx =$

A. $\frac{5}{2}$

B. $2 - \ln 3 + \ln 4$

C. $4 \ln \frac{3}{2} - 1$

D. $1 + 2 \ln 2$

E. $3 - 4 \ln \frac{1}{2}$

$$\begin{array}{r} -1 \\ x+2 \overline{) -x+2} \\ \underline{-x-2} \\ 4 \end{array}$$

so $\frac{2-x}{2+x} = -1 + \frac{4}{x+2}$

$$\int_0^1 \frac{2-x}{2+x} dx = \left[-x + 4\ln|x+2|\right]_0^1$$

$$= -1 + 4\ln 3 - 4\ln 2$$

$$= -1 + 4\ln\left(\tfrac{3}{2}\right)$$

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PART B. Written-Answer Questions

1. *[17 marks]*

If $f(x) = x^3 e^{-x}$ and $f'(x) = x^2(3-x)e^{-x}$ and $f''(x) = x(x^2-6x+6)e^{-x}$ find

[3] (a) any horizontal and vertical asymptote(s) (justifying your answer)

There are $\boxed{\text{no V.A.}}$ since x^3e^{-x} is cont. everywhere.

$\lim_{x\to-\infty} x^3e^{-x} = -\infty$ so no H.A. as $x \to -\infty$

$\lim_{x\to\infty} x^3e^{-x} = \lim_{x\to\infty} \frac{x^3}{e^x} = \lim_{x\to\infty} \frac{3x^2}{e^x} = \lim_{x\to\infty} \frac{6x}{e^x} = \lim_{x\to\infty} \frac{6}{e^x} = 0$

so $\boxed{y = 0 \text{ is a H.A. as } x \to \infty}$

[4] (b) where f is increasing, decreasing and all relative extrema

crit pts. at $x=0$ and $x=3$

	x	f'	f
$(-\infty, 0)$	-10	+	increasing
$(0, 3)$	1	+	increasing
$(3, \infty)$	4	−	decreasing

f is $\boxed{\text{increasing on } (-\infty, 3)}$ and $\boxed{\text{decreasing on } (3,\infty)}$

There is a $\boxed{\text{maximum}}$ (relative and absolute) $\boxed{\text{at } x=3}$.

Note: no relative extremum at $x=0$: f' did ~~not~~ not change sign. (and $f''(0)=0$).

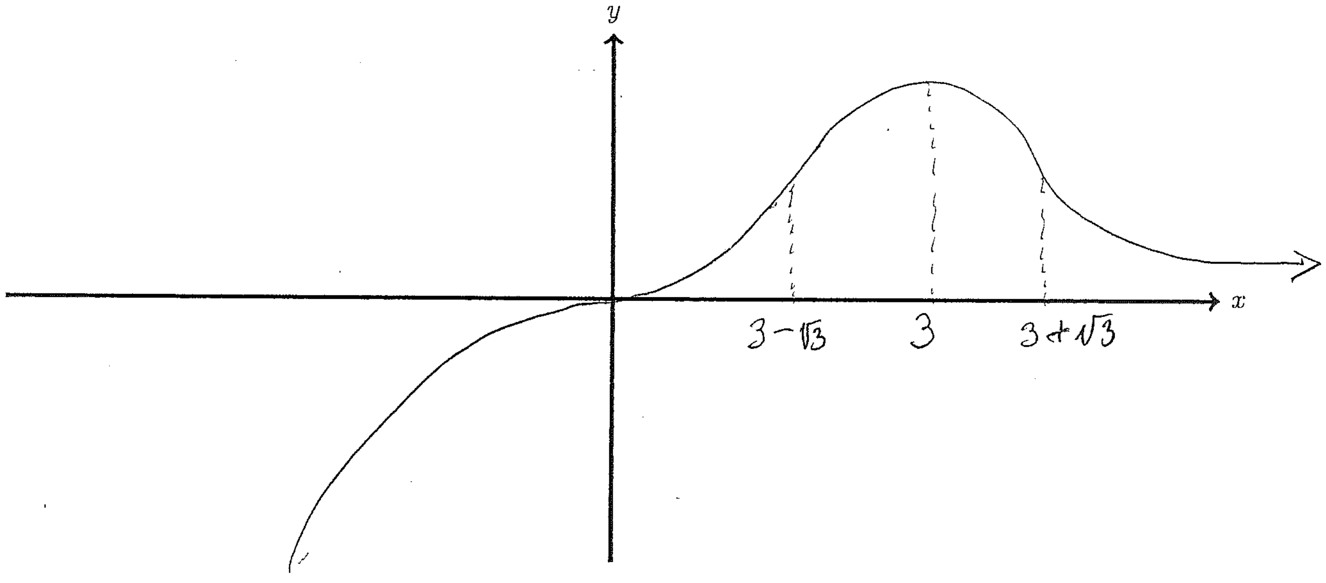
[4] (c) where f is concave up, concave down, and all inflection points

$f''=0$ at $x=0$ and $x = \frac{6 \pm \sqrt{36-24}}{2} = \frac{6 \pm \sqrt{12}}{2} = 3 \pm \sqrt{3}$

	x	f''	f
$(-\infty, 0)$	-1	−	conc down
$(0, 3-\sqrt{3})$	1	+	conc up
$(3-\sqrt{3}, 3+\sqrt{3})$	3	−	conc down
$(3+\sqrt{3}, \infty)$	10^{100}	+	conc up

$\boxed{x = 0,\ 3-\sqrt{3},\ 3+\sqrt{3} \text{ are all p.o.i.}}$

[6] (d) sketch the graph of $y = f(x)$ on the axes provided



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2. *[15 marks]*

Remember to justify your answers

[5] (a) Find and classify the relative extrema of $f(x) = x^3 - 12x^2 + 36x + 1$

$$f'(x) = 3x^2 - 24x + 36 = 3(x^2 - 8x + 12) = 3(x-6)(x-2)$$

crit pts are $x = 2$ and $x = 6$.

Method 1:

	f'	f
$(-\infty, 2)$	+	inc
$(2, 6)$	−	dec
$(6, \infty)$	+	inc

$\wedge$ $\vee$

$\boxed{\begin{array}{l} x=2 \text{ is a local max} \\ x=6 \text{ is a local min} \end{array}}$

Method 2: $f''(x) = 6x - 24$ $\quad f''(2) = -12 < 0$ $\cap$ local max

$f''(6) = 12 > 0$ $\cup$ local min

same answer

[5] (b) The total cost $c(q)$, in dollars, to a company for producing q hundreds of units of widgets is given by

$$c(q) = q^3 - 12q^2 + 36q + 1$$

Find all the values of q that minimize cost and all the values of q that maximize cost when $1 \le q \le 8$.

Since $c(q)$ is cont. the minimum and maximum values must be at $q=1$, $q=8$ or at crit pts in the interior; $q=2$ and $q=6$,

$C(1) = 26$

$C(2) = 33$

$C(6) = 1$

$C(8) = 33$

$\boxed{\begin{array}{l} \text{The min is at } q=6 \\ \text{The max is at } q=2 \text{ and also at } q=8 \end{array}}$

[5] (c) The company from part (b) sells widgets in bulk at \$15 per hundred units. Find the quantity at which profit is maximized when $1 \le q \le 8$.

If π is profit

$$\pi(q) = 15q - C(q)$$
$$= 15q - [q^3 - 12q^2 + 36q + 1]$$
$$= -q^3 + 12q^2 - 21q - 1$$

$$\frac{d\pi}{dq} = -3q^2 + 24q - 21 = -3(q^2 - 8q + 7)$$
$$= -3(q-7)(q-1)$$

Crit at $q = 7$ and $q = 1$

For the same reason as in (b) check at $q=1$, $q=7$, $q=8$

$\pi(1) = 15 - C(1) = 15 - 26 = -11$

$\pi(8) = 120 - C(8) = 120 - 33 = 87$

$\pi(7) = 99$ $\boxed{\text{max at } q = 7}$

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3. *[12 marks]*

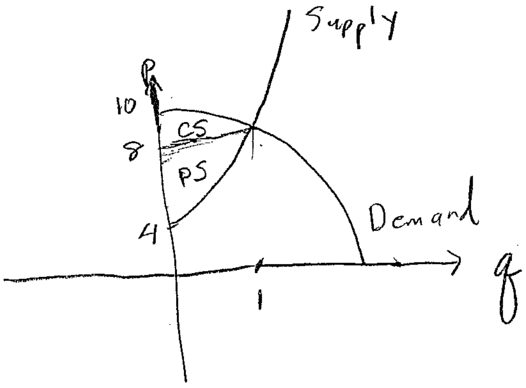
For some product, the demand function is given by $p = -2q^2 + 10$, $(q < \sqrt{5})$, and the supply function by $p = 4q^2 + 4$. Determine the consumers' surplus and the producers' surplus for the product at market equilibrium.

Equilibrium: Demand = Supply

$$-2q^2+10 = 4q^2+4$$

$$6 = 6q^2$$

$q_0 = 1$ so $p_0 = 8$ $\boxed{(p_0,q_0)=(8,1)}$



$$CS = \int_0^1 [(-2q^2+10)-8]\,dq = \int_0^1 2-2q^2\,dq$$

$$= \left(2q-\frac{2q^3}{3}\right)_0^1 = 2-\frac{2}{3} = \boxed{\frac{4}{3} = CS}$$

$$PS = \int_0^1 [8-(4q^2+4)]\,dq$$

$$= \int_0^1 (4-4q^2)\,dq = \left(4q-\frac{4q^3}{3}\right)_0^1 = 4-\frac{4}{3} = \boxed{\frac{8}{3} = PS}$$

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4. *[16 marks]*

Evaluate

[8] (a) $\int_2^3 \frac{dx}{x^3 - x}$

$$\frac{1}{x^3-x} = \frac{1}{x(x^2-1)} = \frac{1}{x(x+1)(x-1)} = \frac{A}{x} + \frac{B}{x+1} + \frac{C}{x-1}$$

$$A(x+1)(x-1) + Bx(x-1) + Cx(x+1) = 1$$

$x = 1$ gives $2C = 1$ $C = \frac{1}{2}$

$x = -1$ gives $2B = 1$ $B = \frac{1}{2}$

$x = 0$ gives $-A = 1$ $A = -1$

$$\int_2^3 \frac{dx}{x^3-x} = \int_2^3 \left(\frac{1}{2}\left[\frac{1}{x+1} + \frac{1}{x-1}\right] - \frac{1}{x}\right) dx$$

$$= \left(\frac{1}{2}\left[\ln|x+1| + \ln|x-1|\right] - \ln x\right)_2^3$$

$$= \left(\frac{1}{2}\left[\ln|x^2-1|\right] - \ln x\right)_2^3$$

$$= \left(\frac{1}{2}\ln 8 - \ln 3\right) - \left(\frac{1}{2}\ln 3 - \ln 2\right)$$

$$= \boxed{\frac{1}{2}\ln 8 - \frac{3}{2}\ln 3 + \ln 2}$$

or since $8 = 2^3$
$\ln 8 = 3 \ln 2$

$$= \frac{5}{2}\ln 2 - \frac{3}{2}\ln 3 = \frac{1}{2}\ln\left(\frac{32}{27}\right)$$

$$\approx .085$$

[8] (b) $\int \frac{dx}{1+\sqrt{x}}$ (this one may be a little tricky)

The substitution $u = 1 + \sqrt{x}$ will do it.

$$du = \frac{1}{2\sqrt{x}}\,dx \qquad u - 1 = \sqrt{x}$$

$$2\sqrt{x}\,du = dx$$

$$2(u-1)\,du = dx$$

$$\int \frac{dx}{1+\sqrt{x}} = \int \frac{2(u-1)\,du}{u} = 2\int\left(1 - \frac{1}{u}\right) du$$

$$= 2\left[u - \ln|u|\right] + C$$

$$= \boxed{2\left[1 + \sqrt{x} - \ln(1+\sqrt{x})\right] + C}$$

Also good is $2\left[\sqrt{x} - \ln(1+\sqrt{x})\right] + C$ since it differs from the previous answer by a constant. This is what you would get if you ~~started~~ with $u = \sqrt{x}$ which will also work.