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Department of Mathematics
University of Toronto

Tuesday, March 3, 2015, 6:10-8:00 PM
MAT 133Y TERM TEST #3

Calculus and Linear Algebra for Commerce
Duration: 1 hour 50 minutes

Aids Allowed: A non-graphing calculator, with empty memory, to be supplied by student.

Instructions: Fill in the information on this page, and make sure your test booklet contains 10 pages. In addition, you should have a **multiple-choice answer sheet**, on which you should fill in your name, number, tutorial time, tutorial room, and tutor's name.

This test consists of 10 multiple choice questions, and 4 written-answer questions.
For the **multiple choice questions** you can do your rough work in the test booklet, but you must record your answer by circling the appropriate letter **on the answer sheet** with your pencil. Each correct answer is worth 4 marks; a question left blank, or an incorrect answer, or two answers for the same question is worth 0. For the **written-answer questions**, present your solutions in the space provided. The value of each written-answer question is indicated beside it.

ENCLOSE YOUR FINAL ANSWER IN A BOX AND WRITE IT IN INK.

TOTAL MARKS: 100

FAMILY NAME: ______________________

GIVEN NAME: ______________________

STUDENT NO: ______________________

SIGNATURE: ______________________

TUTORIAL TIME and ROOM: ______________________

REGCODE and TIMECODE: ______________________

T.A.'S NAME: ______________________

Regcode	Timecode	Room	**Regcode**	Timecode	Room
T0101A	M9A	SS1086	T0501B	W3B	UC261
T0101B	M9B	SS1088	T0601A	R4A	BA2155
T0101C	M9C	SS2128	T0601B	R4B	UC261
T0201A	M3A	UC261	T0601C	R4C	UC330
T0201B	M3B	BL112	T0601D	R4D	BL114
T0201C	M3C	BL114	T0701A	F2A	SS1070
T0201D	M3D	BF215	T0701B	F2B	SS1074
T0301A	T3A	BA2139	T0701C	F2C	SS1083
T0301B	T3B	BA2159	T0701D	F2D	SS2106
T0301C	T3C	WW126	T0801A	F3A	SS1070
T0301D	T3D	AB114	T0801B	F3B	SS1074
T0401A	W9A	LM155	T5101A	M5A	BA1210
T0401B	W9B	LM157	T5101B	M5B	BL114
T0401C	W9C	MP118	T5201A	M6A	SS1088
T0501A	W3A	SS1087			

FOR MARKER ONLY	
Multiple Choice	
B1	
B2	
B3	
B4	
TOTAL	

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PART A. Multiple Choice

1. *[4 marks]*

If $-\frac{1}{2} \leq x \leq \frac{1}{2}$, the function $y = f(x) = \dfrac{1+x^2}{1-x^2}$ has

A. no absolute maximum value.

B. an absolute maximum value of $y = \frac{32}{9}$.

C. an absolute maximum value of $y = 1$.

D. an absolute maximum value of $y = \frac{5}{3}$.

E. an absolute maximum value of $y = \infty$.

$f'(x) = \dfrac{2x(1-x^2) + 2x(1-x^2)}{(1-x^2)^2}$

$= \dfrac{4x}{(1-x^2)^2}$

In $[-\frac{1}{2}, \frac{1}{2}]$, crit. at $x = 0$ only.

Cont. on $[-\frac{1}{2}, \frac{1}{2}]$, so must have absolute max at $x = \frac{1}{2}, -\frac{1}{2}$ or 0.

$f(\frac{1}{2}) = f(-\frac{1}{2}) = \dfrac{1+\frac{1}{4}}{1-\frac{1}{4}} = \frac{5}{3}$

$f(0) = 1$

$\frac{5}{3} > 1$ so $y = \frac{5}{3}$ D

2. *[4 marks]*

The graph of the curve $y = x^3e^{-x}$ has

A. no points of inflection.

B. 1 point of inflection.

C. 2 points of inflection.

D. 3 points of inflection.

E. 4 points of inflection.

$y' = 3x^2e^{-x} - x^3e^{-x}$

$y'' = 6xe^{-x} - 3x^2e^{-x} - 3x^2e^{-x} + x^3e^{-x}$

$= xe^{-x}(6 - 6x + x^2)$

$y'' = 0$ at $x = 0$ and

$x = \dfrac{6 \pm \sqrt{36-24}}{2} = 3 \pm \sqrt{3}$

	f''	conc.
$(-\infty, 0)$	$-$	down
$(0, 3-\sqrt{3})$	$+$	up
$(3-\sqrt{3}, 3+\sqrt{3})$	$-$	down
$(3+\sqrt{3}, \infty)$	$+$	up

$x = -1$

$x = 1$

$x = 3$

10^{100}

All 3 roots of y'' are p.o.i.

D

A

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3. *[4 marks]*

The marginal revenue function for a product is given by $\frac{dr}{dq} = 1000 - 3q^2$ where r is revenue in dollars and q is the number of units sold. Assuming that revenue is 0 if no units are sold, how many units will be sold if the unit price is \$100?

A. 9

B. 33

C. 30

D. 8

E. 17

$r = 1000q - q^3 + K$

$0 = 0 + K$ so $K = 0$

$pq = r = 1000q - q^3$

$p = 1000 - q^2$

$100 = 1000 - q^2$

$q^2 = 900$

$q = 30$ C

4. *[4 marks]*

If $\int_1^2 f(x)\,dx = -6$ and $\int_5^2 f(x)\,dx = 4$, then $\int_1^5 f(x)\,dx =$

A. -10

B. -2

C. 2

D. 10

E. $-\frac{3}{2}$

$\int_1^5 = \int_1^2 + \int_2^5 = \int_1^2 - \int_5^2$

$= -6 - 4$

$= -10$ A

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5. *[4 marks]*

$\int_1^4 \sqrt{x}\,dx =$ $\int_1^4 x^{\frac{1}{2}}\,dx = \frac{2}{3}x^{3/2}\Big|_1^4 = \frac{2}{3}\left[4^{3/2} - 1\right]$

$= \frac{2}{3}\cdot 7 = \frac{14}{3}$ E

A. $\frac{21}{2}$

B. $\frac{2}{3}$

C. 1

D. -1

E. $\frac{14}{3}$

6. *[4 marks]*

If $f(x) = \int_0^x e^{\sqrt{3t+1}}\,dt$, then $f'(8) =$

$f'(x) = e^{\sqrt{3x+1}}$

$f'(8) = e^{\sqrt{25}} = e^5$ E

A. $e^5 - e$

B. $\frac{3}{10}(e^5 - e)$

C. $\frac{3}{10}e^5$

D. $3e^5$

E. e^5

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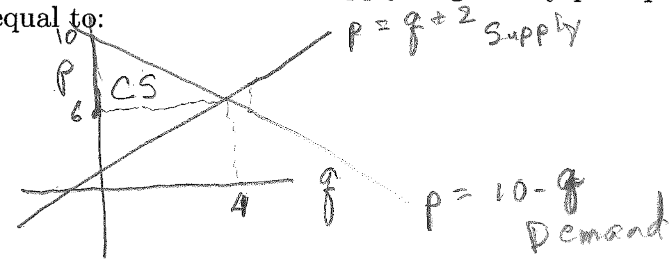
7. *[4 marks]*

If demand is given by $p = 10 - q$ and supply is given by $p = q + 2$, the consumers' surplus (CS) is equal to:



$10 - q = q + 2$

$\left.\begin{array}{l} q_0 = 4 \\ p_0 = 6 \end{array}\right\}$ equilibrium

A. 18

B. 16

C. 32

D. 24

E. 8

$$CS = \int_0^4 [(10-q) - 6]\,dq$$

$$= \int_0^4 (4-q)\,dq = \left[4q - \frac{q^2}{2}\right]_0^4 = 8 \quad \text{E}$$

or more simply, the area of the CS triangle is $\frac{1}{2}bh = \frac{1}{2} \cdot 4 \cdot (10-6) = 8$

8. *[4 marks]*

The average value of $f(x) = x^{-2}$ on the interval $[\frac{1}{7}, 1]$ is

A. $\frac{36}{7}$

B. 7

C. $\frac{28}{3}$

D. 6

E. 8

$$\bar{f} = \frac{1}{1 - \frac{1}{7}} \int_{\frac{1}{7}}^{1} \frac{1}{x^2}\,dx$$

$$= \frac{7}{6}\left(-\frac{1}{x}\right)\Big|_{\frac{1}{7}}^{1} = \frac{7}{6}\left(-1 - \left(-\frac{1}{\frac{1}{7}}\right)\right)$$

$$= \frac{7}{6} \cdot 6 = 7$$

B

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9. *[4 marks]*

A continuous cash-flow of $10,000 per year for 10 years at 4% per year compounded continuously has a present value closest to

A. $67,032

B. $81,109

C. $82,420

D. $122,956

E. $167,580

$$P.V. \approx \int_0^{10} 10{,}000\, e^{-.04t}\, dt$$

$$= \frac{10{,}000}{-.04} e^{-.04t} \Big|_0^{10}$$

$$= 250{,}000\,(1 - e^{-.4})$$

$$\approx 82{,}419.99 \quad C$$

10. *[4 marks]*

$$\int_{-1}^{0} \left[\frac{1}{x+2} + 2^x\right] dx = \left[\ln|x+2| + \frac{2^x}{\ln 2}\right]_{-1}^{0}$$

A. $\ln 2 + \frac{1}{2\ln 2}$

B. $\frac{3}{2}\ln 2$

C. $1 + \frac{1}{2\ln 2}$

D. $\ln 2 + \frac{1}{\ln 2}$

E. $1 + 2\ln 2$

$$= \left(\ln 2 + \frac{1}{\ln 2}\right) - \left(\ln 1 + \frac{1}{2\ln 2}\right)$$

$$= \ln 2 + \frac{1}{2\ln 2} \quad A$$

A

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PART B. Written-Answer Questions

1. *[16 marks]*

Given: $f(x) = \dfrac{e^x}{x^3}$

$f'(x) = e^x x^{-4}(x-3)$

and $f''(x) = e^x x^{-5}(x^2 - 6x + 12)$

[3] (a) find and justify all vertical and horizontal asymptotes.

V.A: $\lim_{x\to 0^+} \frac{e^x}{x} = \infty$ $\quad \lim_{x\to 0^-} \frac{e^x}{x} = -\infty$ $\quad \boxed{\text{V.A. at } x=0}$

H.A.: $\lim_{x\to\infty} \frac{e^x}{x^3} = \lim_{x\to\infty} \frac{e^x}{3x^2} = \lim_{x\to\infty} \frac{e^x}{6x} = \lim_{x\to\infty} \frac{e^x}{6} = \infty$ $\quad \boxed{\text{no H.A. at } +\infty}$

At $-\infty$ $e^x \to 0$ and $x^3 \to -\infty$ so $\frac{e^x}{x^3} \to 0$. $\boxed{\text{H.A. is } y=0 \text{ at } -\infty}$

[3] (b) find where f is increasing and decreasing and any relative maxima and/or minima (with explanation).

	f'	f	
$(-\infty, 0)$	$-$	decreasing	$(x<3$ and $e^{-x}/x^4 > 0)$
$(0, 3)$	$-$	decreasing	"
$(3, \infty)$	$+$	increasing	$(x>3)$

$\boxed{x=3 \text{ is a local min}}$

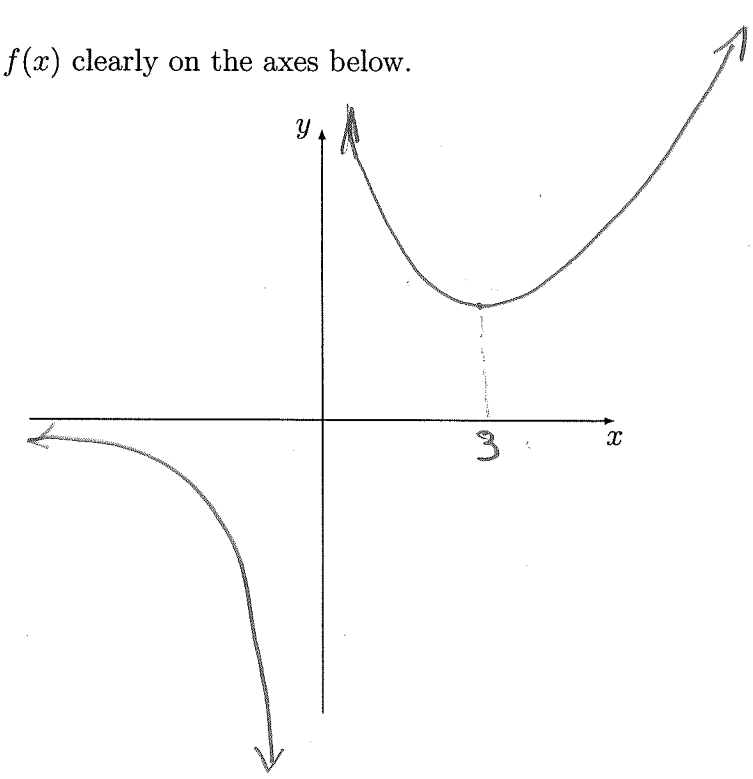
[3] (c) find where f is concave upward and concave downward and any inflection points (with explanation).

$x^2-6x+12$ has no real roots $(6^2-4\cdot 12 < 0)$.
f'' is never 0 and only fails to exist at $x=0$ which is not a point on the curve.

$\boxed{\text{No p.o.i}}$

	f''	conc	
$(-\infty, 0)$	$-$	down	$x=-1$ e.g
$(0, \infty)$	$+$	up	$x=1$ e.g.

[7] (d) graph $y = f(x)$ clearly on the axes below.



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2. *[14 marks]*
A company finds that to sell and produce q units of its product it must set its unit price (to the customer) at $100e^{-.01q}$ dollars, while it costs a total of $400 - 300e^{-.01q}$ dollars to produce the q units.

[7] (a) Find the number of units the company should produce to maximize its profit.

$$\Pi = r - C = pq - C = 100qe^{-.01q} - (400 - 300e^{-.01q})$$

$$\frac{d\Pi}{dq} = 100e^{-.01q} - qe^{-.01q} - 3e^{-.01q}$$

$$= e^{-.01q}(97 - q) = 0 \text{ when } \boxed{q = 97}$$

$0 < q$

[5] (b) Justify your answer to (a).

Note that $\frac{d\Pi}{dq} > 0$ when $q < 97$ so Π is increasing from $q=0$ to $q=97$

but $\frac{d\Pi}{dq} < 0$ for $q > 97$, so Π is decreasing for ever afterwards.

While it is true that $\frac{d^2\Pi}{dq^2} < 0$ at $q = 97$, this only implies a <u>local</u> max. (and for q big enough, the profit curve becomes concave up).

[2] (c) To the nearest dollar, what unit price should the company charge its customers to maximize its profit?

$$p = 100e^{-.97} \approx \$37.91$$

$$\boxed{\$38}$$

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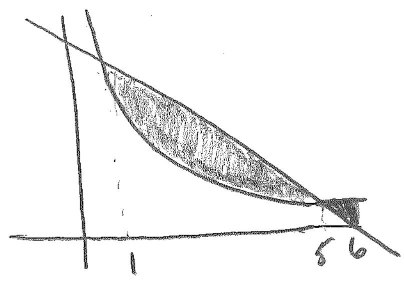
3. *[14 marks]*
Write the area of the region(s) between $y = \frac{5}{x}$ and $y = 6 - x$ from $x = 1$ to $x = 6$ as (a) definite integral(s) and evaluate the integral(s). (Leave your answer in exact form rather than as a decimal number.)



Intersection

$$\frac{5}{x} = 6 - x$$
$$5 = 6x - x^2$$
$$x^2 - 6x + 5 = 0$$
$$(x-1)(x-5) = 0$$
$$x = 1, \quad x = 5$$

$$\text{Area} = \int_1^5 \left[(6-x) - \frac{5}{x}\right] dx + \int_5^6 \left[\frac{5}{x} - (6-x)\right] dx$$

$$= \left[6x - \frac{x^2}{2} - 5\ln|x|\right]_1^5 + \left[5\ln|x| - 6x + \frac{x^2}{2}\right]_5^6$$

$$= \left[\left(30 - \frac{25}{2} - 5\ln 5\right) - \left(6 - \frac{1}{2}\right)\right] + \left[\left(5\ln 6 - 36 + 18\right) - \left(5\ln 5 - 30 + \frac{25}{2}\right)\right]$$

$$= \left[12 - 5\ln 5\right] + \left[-\frac{1}{2} + 5\ln 6 - 5\ln 5\right]$$

$$= \boxed{\frac{23}{2} - 10\ln 5 + 5\ln 6}$$

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4. *[16 marks]* Evaluate the following integrals

[5] (a) $\int_0^2 xe^{x/2}\,dx$

$u = x \quad dv = e^{\frac{x}{2}}dx$

$du = dx \quad v = 2e^{x/2}$

$= 2xe^{\frac{x}{2}}\Big|_0^2 - \int_0^2 2e^{\frac{x}{2}}dx$

$= (4e - 0) - 4e^{\frac{x}{2}}\Big|_0^2$

$= 4e - 4(e-1) = \boxed{4}$

[5] (b) $\int \frac{x}{2x+3}\,dx$

$$2x+3 \,\overline{)\, x \,}\ \ \frac{1}{2}$$
$$x + \frac{3}{2}$$
$$-3/2$$

$= \int \left[\frac{1}{2} - \frac{3}{2}\cdot\frac{1}{2x+3}\right]dx$

$= \frac{x}{2} - \frac{3}{2}\frac{\ln|2x+3|}{2} + C = \boxed{\frac{x}{2} - \frac{3}{4}\ln|2x+3| + C}$

Alternatively: $u = 2x+3 \quad du = 2dx \quad x = \frac{u-3}{2}$

$\int \frac{x}{2x+3}dx = \int \frac{u-3}{2u}\cdot\frac{1}{2}du = \frac{1}{4}\int\frac{u-3}{u}du = \frac{1}{4}\int\left(1 - \frac{3}{u}\right)du = \frac{1}{4}(u - 3\ln|u|) + C$

$= \boxed{\frac{1}{4}(2x+3) - \frac{3}{4}\ln|2x+3| + C} = \frac{x}{2} - \frac{3}{4}\ln|2x+3| + \frac{3}{4} + C$

differing by a constant from the first answer.

[6] (c) $\int \frac{11-3x}{x^2+x-6}\,dx$

$\frac{11-3x}{x^2+x-6} = \frac{A}{x+3} + \frac{B}{x-2}$

$A(x-2) + B(x+3) = 11-3x$

$x = -3 \quad -5A = 20 \quad A = -4$

$x = 2 \quad 5B = 5 \quad B = 1$

$\int \frac{11-3x}{x^2+x-6}dx = \int\left[-\frac{4}{x+3} + \frac{1}{x-2}\right]dx$

$= \boxed{-4\ln|x+3| + \ln|x-2| + C}$