Soln A

Department of Mathematics
University of Toronto

Tuesday, March 4, 2014, 6:10-8:00 PM

MAT 133Y TERM TEST #3

Calculus and Linear Algebra for Commerce
Duration: 1 hour 50 minutes

Aids Allowed: A non-graphing calculator, with empty memory, to be supplied by student.

Instructions: Fill in the information on this page, and make sure your test booklet contains 10 pages. In addition, you should have a **multiple-choice answer sheet**, on which you should fill in your name, number, tutorial time, tutorial room, and tutor's name.

This test consists of 10 multiple choice questions, and 4 written-answer questions.
For the **multiple choice questions** you can do your rough work in the test booklet, but you must record your answer by circling the appropriate letter **on the answer sheet** with your pencil. Each correct answer is worth 4 marks; a question left blank, or an incorrect answer, or two answers for the same question is worth 0. For the **written-answer questions**, present your solutions in the space provided. The value of each written-answer question is indicated beside it.

ENCLOSE YOUR FINAL ANSWER IN A BOX AND WRITE IT IN INK.

TOTAL MARKS: 100

FAMILY NAME: ______________________

GIVEN NAME: ______________________

STUDENT NO: ______________________

SIGNATURE: ______________________

TUTORIAL TIME and ROOM: ______________________

REGCODE and TIMECODE: ______________________

T.A.'S NAME: ______________________

Regcode	Timecode	Room	Regcode	Timecode	Room
T0101A	M9A	SS1084	T0501B	W3B	UCA101
T0101B	M9B	SS1086	T0601A	R4A	BA1210
T0101C	M9C	SS1080	T0601B	R4B	BA1220
T0201A	M3A	MP118	T0601C	R4C	AP120
T0201B	M3B	UC52	T0601D	R4D	GB120
T0201C	M3C	UC152	T0701A	F2A	LM157
T0201D	M3D	GB221	T0701B	F2B	BF215
T0301A	T3A	RW143	T0701C	F2C	MP137
T0301B	T3B	BA1210	T0701D	F2D	SS2127
T0301C	T3C	AP120	T0801A	F3A	BF215
T0301D	T3D	WW126	T0801B	F3B	MP118
T0401A	W9A	SS1084	T5101A	M5A	SS2105
T0401B	W9B	SS1088	T5101B	M5B	UC87
T0401C	W9C	LM155	T5201A	M6A	LM162
T0501A	W3A	UC85			

FOR MARKER ONLY	
Multiple Choice	
B1	
B2	
B3	
B4	
TOTAL	

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PART A. Multiple Choice

1. *[4 marks]*

The graph of $y = \dfrac{x+5}{|x|-2}$ has

V.A. $x = 2$ and $x = -2$

H.A. : $\lim_{x\to\infty} y = \lim_{x\to\infty} \frac{x+5}{x-2} = 1$

$\lim_{x\to-\infty} y = \lim_{x\to-\infty} \frac{x+5}{-x-2} = -1$

H.A. $y = 1$ and $y = -1$

D

A. one vertical asymptote and one horizontal asymptote.

B. two vertical asymptotes and one horizontal asymptote.

C. one vertical asymptote and two horizontal asymptotes.

D. two vertical asymptotes and two horizontal asymptotes.

E. two vertical asymptotes and no horizontal asymptotes.

2. *[4 marks]*

Let $f(x) = (3x+1)e^x$. On the interval $(-1, \infty)$, $f(x)$

A. is increasing and concave up.

B. is increasing and concave down.

C. is decreasing and concave up.

D. is decreasing and concave down.

E. switches concavity.

$f'(x) = 3e^x + (3x+1)e^x$
$= (3x+4)e^x > 0$ on $(-\frac{4}{3}, \infty)$

So $\boxed{f \text{ is increasing on } (-1, \infty)}$

$f''(x) = 3e^x + (3x+4)e^x$
$= (3x+7)e^x > 0$ on $(-\frac{7}{3}, \infty)$

So $\boxed{f \text{ is concave up on } (-1, \infty)}$

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3. *[4 marks]*

If $f''(x) = 6x - \dfrac{1}{x^2}$ and $f'(1) = 4$ and $f(1) = 2$ then $f(e) =$

A. $e^3 + 2e - 2$

B. $3e^2 + \dfrac{1}{e}$

C. $e^3 + 1$

D. e^3

E. $e^3 + 2$

$f'(x) = 3x^2 + \frac{1}{x} + C$

$4 = 3 + 1 + C \Rightarrow C = 0$

$f'(x) = 3x^2 + \frac{1}{x}$

$f(x) = x^3 + \ln|x| + C$

$2 = 1 + \ln 1 + C \Rightarrow C = 1$

$f(x) = x^3 + \ln|x| + 1$

$f(e) = e^3 + \ln e + 1 = e^3 + 2$

E

4. *[4 marks]*

$$\int \frac{4x^2+1}{2x-1}\,dx =$$

A. $x^2 + x + 2\ln|2x-1| + C$

B. $x^2 + x + C$

C. $x^2 - x + C$

D. $x^2 + x + \ln|2x-1| + C$

E. $2x + 1 + \dfrac{2}{2x-1} + C$

$$\begin{array}{r} 2x+1 \\ 2x-1\,\overline{)\,4x^2+1} \\ \underline{4x^2-2x} \\ 2x+1 \\ \underline{2x-1} \\ 2 \end{array}$$

$\int \frac{4x^2+1}{2x-1}\,dx = \int \left(2x+1 + \frac{2}{2x-1}\right) dx$

$= x^2 + x + \ln|2x-1| + C$

D

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5. *[4 marks]*

$$\int_{-1}^{0} 5^{(x^3-3x)}(x^2-1)\,dx =$$

Hint: Let $u = x^3 - 3x$.

A. $-\frac{50}{3}$

B. $-\frac{24}{\ln 5}$

C. $-\frac{8}{\ln 5}$

D. $\frac{8}{\ln 5}$

E. -8

Let $u = x^3 - 3x$

$du = (3x^2-3)\,dx$

$\frac{du}{3} = (x^2-1)\,dx$

$$\int_{-1}^{0} 5^{x^3-3x}(x^2-1)\,dx = \frac{1}{3}\int_{2}^{0} 5^{u}\,du$$

$$= -\frac{1}{3}\int_{0}^{2} 5^{u}\,du$$

$$= -\frac{1}{3}\left.\frac{5^{u}}{\ln 5}\right|_{0}^{2}$$

$$= -\frac{1}{3\ln 5}(5^2-5^0)$$

$$= -\frac{24}{3\ln 5} = -\frac{8}{\ln 5}$$

C

6. *[4 marks]*

If A and B are real numbers such that $\frac{A}{x-1} + \frac{B}{x+2} = \frac{3x+3}{(x-1)(x+2)}$ for all real x except $x = 1$ and $x = -2$, then $A =$

A. 1

B. -1

C. -2

D. 3

E. 2

$A(x+2) + B(x-1) = 3x+3$

Fast way: $x = 1 \Rightarrow 3A = 6 \Rightarrow A = 2$ E

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7. *[4 marks]*

If $f(x) = \int_2^x e^{t-2} \ln(t^2+1)\, dt$, $f'(3) =$

$f'(x) = e^{x-2} \ln(x^2+1)$

$f'(3) = e \ln 10$ B

A. $e \ln 10 - \ln 5$

B. $e \ln 10$

C. $\frac{3}{5}e + e \ln 10$

D. $\frac{3}{5}e + e \ln 10 - \frac{4}{5} - \ln 5$

E. $\ln 5$

8. *[4 marks]*

The average value of $f(x) = x^3$ on the interval $[-3, 1]$ is

A. -2

B. -5

C. -3

D. -6

E. -4

$\frac{1}{1-(-3)} \int_{-3}^{1} x^3 dx$

$= \frac{1}{4} \frac{x^4}{4} \Big|_{-3}^{1} = \frac{1}{16}(1-81) = -5$

B

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9. *[4 marks]*

$\int_4^\infty \frac{dx}{\sqrt{x}} =$ $\lim_{R\to\infty} \int_4^R \frac{dx}{\sqrt{x}} = \lim_{R\to\infty} 2\sqrt{x}\Big|_4^R$

$= \lim_{R\to\infty} (2\sqrt{R} - 4) \to \infty$ D

A. 1

B. 4

C. $\frac{1}{2}$

D. ∞, that is, the integral diverges

E. 2

10. *[4 marks]*

Beginning at a certain time ($t = 0$, where t is given in years), cash flows continuously into an account at the rate $1,000e^{.04t}$ dollars per year. If the account earns interest at the nominal annual rate of 6% compounded continuously and the cash flow stops at $t = 10$ years then its present value is

A. \$ 9,063.46

B. \$12,295.62

C. \$ 8,242.00

D. \$11,070.14

E. \$ 9,547.32

$\int_0^{10} 1000\, e^{.04t} e^{-.06t}\, dt$

$1000 \int_0^{10} e^{-.02t}\, dt$

$= \frac{1000}{-.02} e^{-.02t}\Big|_0^{10}$

$= 50,000(1 - e^{-.2})$

$= 9063.46$ A

A

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PART B. Written-Answer Questions

1. *[17 marks]*

Let $f(x) = \dfrac{x}{x^2-1}$ $\qquad f'(x) = -\dfrac{(x^2+1)}{(x^2-1)^2}$ $\qquad f''(x) = \dfrac{2x(x^2+3)}{(x^2-1)^3}$

[3] (a) Find all vertical and horizontal asymptotes (if any) of $y = f(x)$.

$\boxed{\text{V.A. } x=-1 \text{ and } x=1}$

H.A. $\lim_{x\to\infty} f(x) = 0 = \lim_{x\to-\infty} f(x)$ $\qquad \boxed{y=0 \text{ is H.A. at } \pm\infty}$

[3] (b) Find and classify all critical points (if any). Find the intervals in which $y = f(x)$ is increasing and/or decreasing.

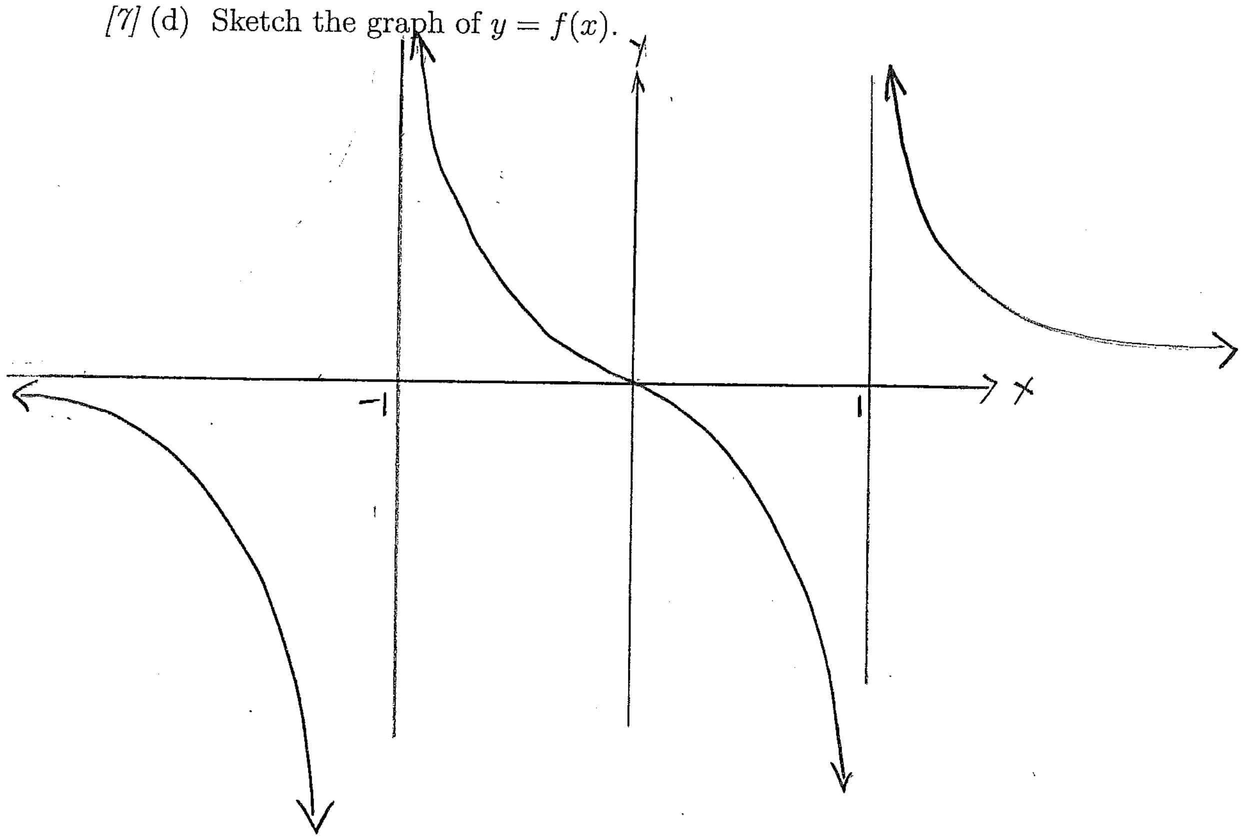
There are no critical points; but f' is not defined where f is not defined, namely at -1 and $+1$.

$\boxed{\begin{array}{l} f'<0 \text{ on } (-\infty,-1),\ (-1,1) \text{ and } (1,\infty) \\ f \text{ is decreasing on each of these intervals} \end{array}}$

[4] (c) Find the points of inflection and the intervals on which $y = f(x)$ is concave up and/or down.

	f''	f	
$(-\infty,-1)$	$-$	conc down	$x=-1$ not on curve
$(-1,0)$	$+$	conc up	$\boxed{x=0 \text{ is a p.o.i.}}$
$(0,1)$	$-$	conc down	$x=1$ not on curve
$(1,\infty)$	$+$	conc up	

[7] (d) Sketch the graph of $y = f(x)$.



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2. *[13 marks]*

The demand function for a product is given by $q = \dfrac{90}{p} - 2$ and the total cost function by $C = \dfrac{q}{20} + 30$where p is measured in thousands of dollars and q in thousands of items.

How much is the maximum profit in this situation and how many items will be sold at what price? [Make sure to include an argument that your answer indeed maximizes profit.]

$$\Pi = r - C = pq - C$$

Soln 1: $q + 2 = \dfrac{90}{p}$

$$p = \frac{90}{q+2}$$

$$\Pi = \frac{90q}{q+2} - \left(\frac{q}{20} + 30\right)$$

$$\frac{d\Pi}{dq} = 90\left[\frac{(q+2)-q}{(q+2)^2}\right] - \frac{1}{20} = \frac{180}{(q+2)^2} - \frac{1}{20}$$

$\dfrac{d\Pi}{dq} = 0$ when $(q+2)^2 = 3600$ $\quad q+2 = 60$ $\quad \boxed{q = 58}$

on $(0,58)$ $\dfrac{d\Pi}{dq} > 0$ on $(58, \infty)$ $\dfrac{d\Pi}{dq} < 0$

$\therefore$ Π is absolute max at $\boxed{q = 58}$. $p = \dfrac{90}{60} = \boxed{1.5 = p}$

and $\Pi = 58 \times 1.5 - \left(\dfrac{58}{20} + 30\right) = \boxed{54.1 \text{ thousand}}$

Soln 2: $C = \dfrac{1}{20}\left(\dfrac{90}{p} - 2\right) + 30 = \dfrac{9}{2p} + 29.9$

$$r = pq = 90 - 2p$$

$$\Pi = 90 - 2p - \left(\frac{9}{2p} + 29.9\right)$$

$\dfrac{d\Pi}{dp} = -2p + \dfrac{9}{2p^2} = 0$ when $p^3 = \dfrac{9}{4}$ $\Rightarrow \boxed{p = 1.5}$

$q = 58$

$\Pi = 54.1$

as before

$= \dfrac{9 - 4p^2}{2p^2} > 0$ when $p < \frac{3}{2}$

and < 0 when $p > \frac{3}{2}$

So Π is absolute max at $p = \frac{3}{2}$

A

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3. *[14 marks]* Write the areas of the regions described below in terms of definite integral(s) without using absolute value signs **but do not not evaluate the definite integral(s).**

[7] (a) The area between $y = x^2(2+x)$ and the x-axis from $x = -3$ to $x = 0$

Inside $(-3,0)$, $y = x^2(2+x)$ crosses the x-axis at $x=-2$ only. So on $[-3,-2]$ $y<0$ (lies below the x-axis); and on $[-2,0]$ $y>0$ (lies above the x-axis),

$$\text{Area} = \int_{-3}^{-2} [0 - x^2(2+x)]\,dx + \int_{-2}^{0} [x^2(2+x) - 0]\,dx$$

or $$-\int_{-3}^{-2} x^2(2+x)\,dx + \int_{-2}^{0} x^2(2+x)\,dx$$

[7] (b) The area of the regions bounded by the line $y = x$ and the curve $y = \dfrac{2x}{x^2+1}$

The curves intersect when

$$x = \frac{2x}{x^2+1} \qquad x(x^2+1) = 2x$$
$$x(x^2-1) = 0$$
$$x = 0, \quad x = \pm 1 \quad \text{i.e. 3 times}$$

The regions lie between $x=-1$ and $x=0$ and between $x=0$ and $x=1$

On $(-1,0)$: the curves cannot cross, so, trying $x=-\frac{1}{2}$,

$y = x = -\frac{1}{2}$ $\quad y = \frac{2x}{x^2+1} = \frac{-1}{\frac{5}{4}} = -\frac{4}{5} < -\frac{1}{2}$

$y = x$ lies above $y = \frac{2x}{x^2+1}$ on $(-1,0)$.

On $(0,1)$ the curves cannot cross, so, trying $x = \frac{1}{2}$,

$y = x = \frac{1}{2}$ $\quad y = \frac{2x}{x^2+1} = \frac{4}{5} > \frac{1}{2}$

$y = \frac{2x}{x^2+1}$ lies above $y = x$ on $(0,1)$

$$\text{Area} = \int_{-1}^{0} \left(x - \frac{2x}{x^2+1}\right)dx + \int_{0}^{1} \left(\frac{2x}{x^2+1} - x\right)dx$$

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4. *[16 marks]* Find the indefinite integrals:

[8] (a) $\int x^2 e^{x/2}\,dx$

$u = x^2 \quad dv = e^{x/2}dx$

$du = 2x\,dx \quad v = 2e^{\frac{x}{2}}$

$= 2x^2e^{x/2} - \int 4xe^{\frac{x}{2}}dx$

Let $u = x \quad dv = e^{x/2}$

$du = dx \quad v = 2e^{x/2}$

$=$

$2x^2e^{x/2} - 4\left[2xe^{x/2} - 2\int e^{x/2}dx\right]$

$= 2x^2e^{x/2} - 8xe^{x/2} + 8 \cdot 2e^{x/2} + C$

$= 2e^{x/2}\left[x^2 - 4x + 8\right] + C$

[8] (b) $\int \frac{x^2 - 2x - 3}{x^2(2x+3)}\,dx$

$\frac{A}{x} + \frac{B}{x^2} + \frac{C}{2x+3} = \frac{x^2-2x-3}{x^2(2x+3)}$

$Ax(2x+3) + B(2x+3) + Cx^2 = x^2 - 2x - 3$

$x = 0$ gives $3B = -3$ $\boxed{B = -1}$

$x = -\frac{3}{2}$ gives $\frac{9}{4}C = \frac{9}{4} + 3 - 3 \Rightarrow \boxed{C = 1}$

$x = -1$ (for example) gives $-A + B + C = 1 + 2 - 3$

$-A - 1 + 1 = 0 \Rightarrow \boxed{A = 0}$

$\int \frac{x^2 - 2x - 3}{x^2(2x+3)}\,dx = \int\left(-\frac{1}{x^2} + \frac{1}{2x+3}\right)dx$

$= \boxed{\frac{1}{x} + \frac{1}{2}\ln|2x+3| + C}$