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Department of Mathematics
University of Toronto

Tuesday, March 5, 2013, 6:10 - 8 PM
MAT 133Y TERM TEST # 3

Calculus and Linear Algebra for Commerce
Duration: 1 hour 50 minutes

Aids Allowed: A non-graphing calculator, with empty memory, to be supplied by student.

Instructions: Fill in the information on this page, and make sure your test booklet contains 10 pages. In addition, you should have a **multiple-choice answer sheet**, on which you should fill in your name, number, tutorial time, tutorial room, and tutor's name.

This test consists of 10 multiple choice questions, and 4 written-answer questions.
For the **multiple choice questions** you can do your rough work in the test booklet, but you must record your answer by circling the appropriate letter **on the answer sheet** with your pencil. Each correct answer is worth 4 marks; a question left blank, or an incorrect answer, or two answers for the same question is worth 0. For the **written-answer questions**, present your solutions in the space provided. The value of each written-answer question is indicated beside it.

ENCLOSE YOUR FINAL ANSWER IN A BOX AND WRITE IT IN INK.

TOTAL MARKS: 100

FAMILY NAME: _____

GIVEN NAME: _____

STUDENT NO: _____

SIGNATURE: _____

TUTORIAL TIME and ROOM: _____

REGCODE and TIMECODE: _____

T.A.'S NAME: _____

Regcode	Timecode	Room	**Regcode**	Timecode	Room
T0101A	M9A	SS1084	T0501C	W3C	UC328
T0101B	M9B	SS1086	T0501D	W3D	SF1101
T0201A	M3A	UC244	T0601A	R4A	MP137
T0201B	M3B	LM123	T0601B	R4B	MS2173
T0201C	M3C	RW142	T0601C	R4C	BA1220
T0201D	M3D	BA1170	T0601D	R4D	BA 1200
T0301A	T3A	RW143	T0701A	F2A	SS1073
T0301B	T3B	WI524	T0701B	F2B	RW229
T0301C	T3C	AB107	T0701C	F2C	RW142
T0401A	W9A	SS1070	T0701D	F2D	MP134
T0401B	W9B	SS1072	T0801A	F3A	SS1087
T0401C	W9C	SS1074	T0801B	F3B	LM 157
T0501A	W3A	SS1087	T5101A	M5A	SS1069
T0501B	W3B	ES B142	T5201A	M6A	LM 158

FOR MARKER ONLY	
Multiple Choice	
B1	
B2	
B3	
B4	
TOTAL	

NAME: _____ STUDENT NO: _____

PART A. Multiple Choice

1. *[4 marks]*

The demand function of a product is $p = 600 - 3q$ and the **average** cost per unit for producing q units is $\bar{c} = q + 200 + \frac{500}{q}$. The maximum profit is

A. 8500

B. 9000

C. 9500

D. 10,000

E. 10,500

$\Pi = R - C = pq - \bar{c}q$ (since $\frac{C}{q} = \bar{c}$)

$= (600 - 3q)q - (q^2 + 200q + 500)$

$= -500 + 400q - 4q^2$

$\frac{d\Pi}{dq} = 400 - 8q = 0$ when $q = 50$

(and $\frac{d^2\Pi}{dq^2} = -8 < 0$ for ~~every~~ q)

Max profit at $q = 50$

$\Pi = -500 + 400 \times 50 - 4 \times 50^2$

$= \boxed{9,500}$ (C)

2. *[4 marks]*

If $f(x) = \frac{1}{6}x^4 + x^3 - 4x^2 + 1$, then f is concave up on the interval(s)

A. $(-\infty, \infty)$

B. $(-\infty, -4)$ and $(1, \infty)$

C. $(-4, 1)$

D. $(-4, \infty)$

E. $(-\infty, -1)$ and $(4, \infty)$

$f'(x) = \frac{4x^3}{6} + 3x^2 - 8x$

$f''(x) = 2x^2 + 6x - 8$

$= 2(x^2 + 3x - 4)$

$= 2(x+4)(x-1)$

	f''	conc
$(-\infty, -4)$	+	up
$(-4, 1)$	−	down
$(1, \infty)$	+	up

(B)

NAME: _____ STUDENT NO: _____

3. *[4 marks]*

If $f''(x) = 28x^{1/3}$, $f(0) = 2$, and $f'(0) = 3$, then $f(1) =$

A. 16

B. 14

C. 17

D. 13

E. 15

$f'(x) = \frac{28 x^{4/3}}{\frac{4}{3}} + C = 21x^{4/3} + C$

$3 = f'(0) = C$

$f'(x) = 21x^{4/3} + 3$

$f(x) = 21x^{7/3} \cdot \frac{3}{7} + 3x + K$

$f(x) = 9x^{7/3} + 3x + K$

$2 = f(0) = K$

$f(x) = 9x^{7/3} + 3x + 2$

$f(1) = 9 + 3 + 2 = \boxed{14}$ (B)

4. *[4 marks]*

If a company's marginal cost is $\frac{dc}{dq} = \frac{100}{(q+1)^2}$ when it is producing q units of a product, and its start-up cost is 50, how much does it cost to produce 24 units?

A. 144

B. 162

C. 140

D. 146

E. 150

$C = \frac{-100}{q+1} + K$ $\qquad C(0) = 50$

$50 = C(0) = -100 + K \qquad K = 150$

$C = \frac{-100}{q+1} + 150$

$C(24) = \frac{-100}{25} + 150 = \boxed{146}$ (D)

NAME: _____ STUDENT NO: _____

5. *[4 marks]*

If $f(x) = \int_0^x \frac{dt}{t^2+1}$, then $f'(2) =$

$f'(x) = \frac{1}{x^2+1}$

$f'(2) = \boxed{\frac{1}{5}}$ (A)

A. $\frac{1}{5}$

B. $\frac{1}{2}$

C. $-\frac{4}{25}$

D. $\frac{2}{5}$

E. $-\frac{2}{5}$

6. *[4 marks]*

$\int_{\ln a}^{\ln b} e^{2x} dx =$

$\frac{e^{2x}}{2}\Big|_{\ln a}^{\ln b} = \frac{e^{2\ln b} - e^{2\ln a}}{2} = \frac{(e^{\ln b})^2 - (e^{\ln a})^2}{2}$

$= \frac{b^2 - a^2}{2}$ (B)

A. $b^2 - a^2$

B. $\frac{b^2 - a^2}{2}$

C. $\sqrt{b} - \sqrt{a}$

D. $\sqrt{\frac{b}{a}}$

E. $\frac{b^2}{a^2}$

NAME: _____ STUDENT NO: _____

7. *[4 marks]*

$\int_0^1 \frac{x+2}{x+1}dx =$

A. $1+\ln\left(\frac{3}{2}\right)$

B. $1+\ln 2$

C. $\ln 3 - \ln 2$

D. 2

E. $1+2\ln 2$

$\frac{x+2}{x+1} = 1+\frac{1}{x+1}$

$\int_0^1 \frac{x+2}{x+1}\,dx = \int_0^1 \left(1+\frac{1}{x+1}\right)dx$

$= [x+\ln|x+1|]_0^1$

$= 1+\ln 2 - 0$

$= 1+\ln 2$ (B)

8. *[4 marks]*

The future value of a continuous annuity for 10 years with interest at the rate of 4% compounded continuously and with payment at time t at the rate of \$100/year is:

A. \$1200.61

B. \$824.20

C. \$857.83

D. \$1229.56

E. \$1205.13

$\int_0^{10} 100e^{(10-t)(.04)}\,dt$

$= \frac{100}{.04}e^{.4}\left(-e^{-.04t}\right)\Big|_0^{10}$

$= 2500e^{.4}(1-e^{-.4t})$

$= 2500(e^{.4}-1)$

$= \boxed{1229.56}$ (D)

NAME: _____ STUDENT NO: _____

9. *[4 marks]*

The average value of $f(x) = \frac{1}{\sqrt{x}}$ on the interval [4, 9] is

A. $\frac{5}{\sqrt{13}}$

B. 2

C. $\frac{2}{5}$

D. $\frac{5}{12}$

E. $\frac{1}{12}$

$\bar{f} = \frac{1}{9-4} \int_4^9 \frac{1}{\sqrt{x}}\,dx$

$= \frac{1}{5} 2\sqrt{x} \Big|_4^9 = \frac{2}{5}(\sqrt{9} - \sqrt{4})$

$= \boxed{\frac{2}{5}}$ (C)

10. *[4 marks]*

$\int_1^{+\infty} \frac{1}{(x+1)^2}dx =$

A. $\frac{1}{4}$

B. $-\frac{1}{2}$

C. $-\frac{1}{4}\ln 4$

D. $\frac{1}{2}$

E. does not exist, because the integral diverges

$\lim_{R\to\infty} \int_1^R \frac{1}{(1+x)^2}dx$

$= \lim_{R\to\infty} \left(-\frac{1}{1+x}\right)_1^R$

$= \lim_{R\to\infty} \left(-\frac{1}{1+R} + \frac{1}{2}\right) = \boxed{\frac{1}{2}}$ (D)

NAME: _____ STUDENT NO: _____

PART B. Written-Answer Questions

1. *[16 marks]*

Let

$$f(x) = \frac{9x}{(x+1)^3}, \quad f'(x) = 9\left(\frac{1-2x}{(x+1)^4}\right), \quad f''(x) = 54\left(\frac{x-1}{(x+1)^5}\right)$$

[2] (a) Find the vertical and horizontal asymptotes of $y = f(x)$.

V.A: $y \to +\infty$ only as $x \to -1$ $\boxed{x=-1 \text{ is V.A}}$

$\lim_{x \to \pm\infty} f(x) = 0$ $\boxed{y=0 \text{ is H.A. at } \pm\infty}$

[3] (b) Find the intervals where f is increasing and decreasing and find all relative maximum and minimum points.

Since $(x+1)^4 > 0$ $f'(x) < 0$ when $2x > 1$, i.e. $x > \frac{1}{2}$

$f'(x) > 0$ " $2x < 1$ i.e. $x < \frac{1}{2}$

	f'	f
$(-\infty, -1)$	+	inc
$(-1, \frac{1}{2})$	+	inc
$(\frac{1}{2}, \infty)$	-	dec

$\wedge$ $x = \frac{1}{2}$ is local max

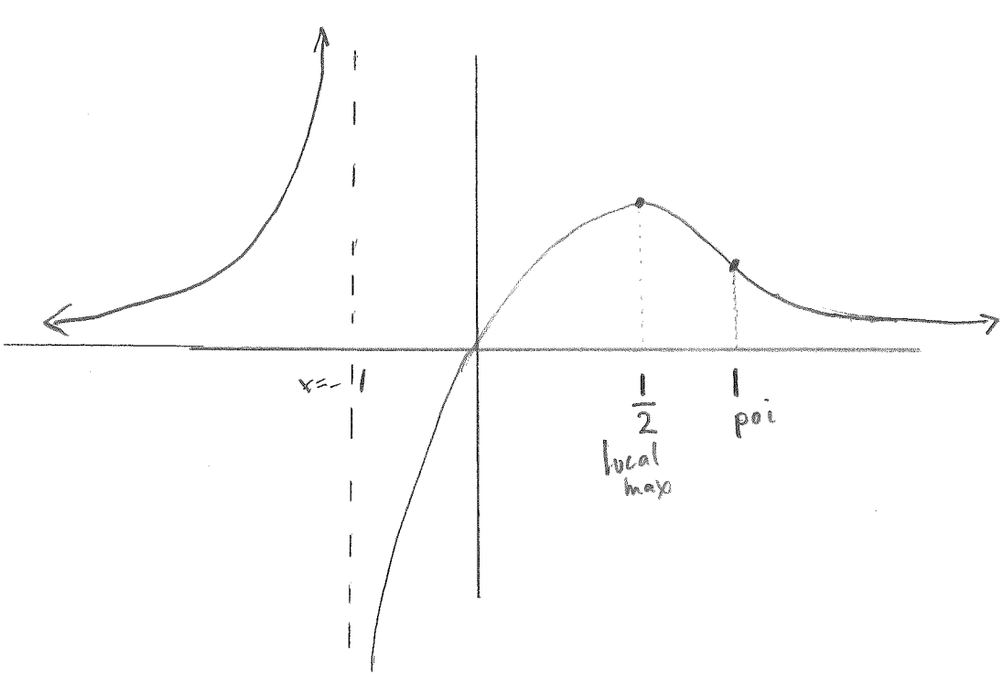
[3] (c) Find the intervals where f is concave up and concave down and find all inflection points.

	f''	f
$(-\infty, -1)$	+	conc up
$(-1, 1)$	-	conc down
$(1, \infty)$	+	conc up

$x=-1$ is not a poi (not a point on the curve)

$x=1$ is a pt. of inflection

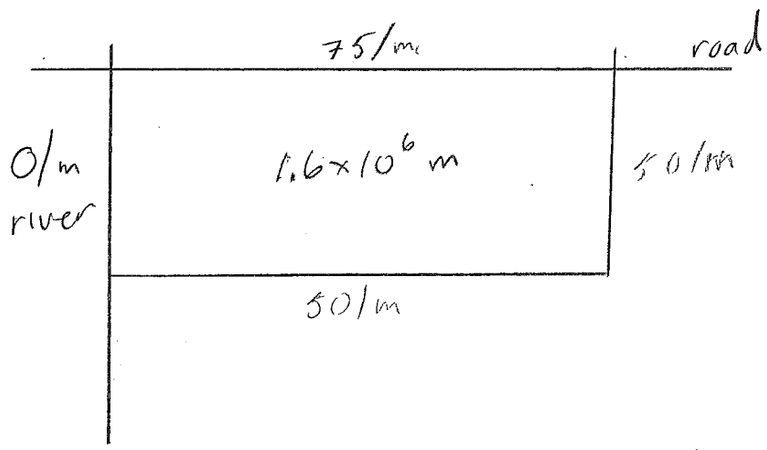
[8] (d) Sketch the graph of f and indicate all relevant information pertaining to f.



NAME: _____ STUDENT NO: _____

2. *[13 marks]*

A farmer needs to enclose a rectangular pasture which has an area of 1,600,000 m^2. One side of the pasture is bounded by a road and the fence along the road costs \$75/m. One side perpendicular to the road is bounded by a river and hence needs no fence. Fences on the other two sides cost \$50/m. What should the dimensions of the pasture be if the farmer wishes to minimize his costs?



Let x be the length of the side along the river
y " " " " " along the road

$$xy = 1.6\times10^6$$

$$C = 50x + 125y = 50x + 125\times\frac{1.6\times10^6}{x} = 50x + \frac{200\times10^6}{x}$$

$x > 0$

$$\frac{dC}{dx} = 50 - \frac{200\times10^6}{x^2} = \frac{50}{x^2}(x^2 - 4\times10^6)$$

$\frac{dC}{dx} = 0$ when $x = 2\times10^3 = 2000$ m (and $y = 800$m)

This is a min because

① $\frac{dC}{dx} < 0$ on $(0, 2000)$ and $\frac{dC}{dx} > 0$ on $(2000, \infty)$
C decreasing C increasing

or ② $\frac{d^2C}{dx^2} = \frac{4\times10^8}{x^3} > 0$ on $(0, \infty)$
C concave up on

or ③ $C \to \infty$ as $x \to 0$ and $x \to \infty$. The only crit point is at $x = 2000$.

3. *[15 marks]*

Find the area of the finite region bounded by the curves $y = x^3 - x^2$ and $y = 2x$. Please express your answer as a single fraction or decimal number.

Intersection: $x^3 - x^2 = 2x$

$x^3 - x^2 - 2x = 0$

$x(x^2 - x - 2) = 0$

$x(x-2)(x+1) = 0$: $\boxed{x = -1,\ x = 0,\ x = 2}$

on $(-1, 0)$, say at $x = -\frac{1}{2}$, $x^3 - x^2 = -\frac{1}{8} - \frac{1}{4} = -\frac{3}{8}$

$2x = -1$

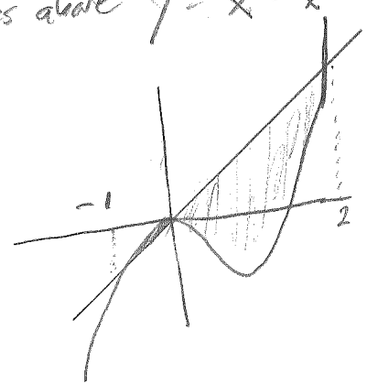
$y = x^3 - x^2$ lies above $y = 2x$.

on $(0, 2)$, say at $x = 1$, $x^3 - x^2 = 0$

$2x = 2$

$y = 2x$ lies above $y = x^3 - x^2$

Unnecessary picture!



$$\text{Area} = \int_{-1}^{0} [(x^3 - x^2) - 2x]\,dx + \int_{0}^{2} [2x - (x^3 - x^2)]\,dx$$

$$= \left(\frac{x^4}{4} - \frac{x^3}{3} - x^2\right)\Big|_{-1}^{0} + \left(x^2 - \frac{x^4}{4} + \frac{x^3}{3}\right)\Big|_{0}^{2}$$

$$= -\left(\frac{1}{4} + \frac{1}{3} - 1\right) + \left(4 - \frac{16}{4} + \frac{8}{3}\right)$$

$$= \frac{5}{12} + \frac{8}{3}$$

$$= \boxed{\frac{37}{12}} \approx \boxed{3.08}$$

NAME: _____ STUDENT NO: _____

4. *[16 marks]*

Find the following integrals

[5] (a) $\int_0^1 x\sqrt{1-x^2}dx$

Let $u = 1-x^2$ $du = -2x\,dx$ $dx = -\frac{1}{2}du$

$= -\frac{1}{2}\int_1^0 \sqrt{u}\,du = \frac{1}{2}\int_0^1 \sqrt{u}\,du = \frac{1}{2}u^{3/2}\cdot\frac{2}{3}\Big|_0^1 = \boxed{\frac{1}{3}}$

[5] (b) $\int_1^e \frac{\ln x}{x^2}dx$ (Hint: by parts)

$u = \ln x$ $dv = \frac{1}{x^2}dx$

$du = \frac{1}{x}dx$ $v = -\frac{1}{x}$

$\int_1^e \frac{\ln x}{x^2}dx = -\frac{1}{x}\ln x\Big|_1^e + \int_1^e \frac{1}{x^2}dx$

$= -\frac{1}{e} - \frac{1}{x}\Big|_1^e = -\frac{1}{e} - \frac{1}{e} + 1$

$= \boxed{1 - \frac{2}{e}}$

[6] (c) $\int \frac{5x^2+13x+8}{x(x+2)^2}dx$

$\frac{5x^2+13x+8}{x(x+2)^2} = \frac{A}{x} + \frac{B}{x+2} + \frac{C}{(x+2)^2}$

$A(x+2)^2 + Bx(x+2) + Cx = 5x^2+13x+8$

$x=0$: $4A = 8$ $A = 2$

$x=-2$ $-2C = 2$ $C = -1$

Coefficient of x^2: $A+B=5$ $2+B=5$ $B=3$

(There are lots of ways to get $B=3$.)

$\int \frac{5x^2+13x+8}{x(x+2)^2}dx = \int\left(\frac{2}{x} + \frac{3}{x+2} - \frac{1}{(x+2)^2}\right)dx$

$= \boxed{2\ln|x| + 3\ln|x+2| + \frac{1}{x+2} + C}$