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Department of Mathematics
University of Toronto

Winter 2012

MAT 133Y TERM TEST #3

Calculus and Linear Algebra for Commerce
Duration: 1 hour 50 minutes

Aids Allowed: A non-graphing calculator, with empty memory, to be supplied by student.

Instructions: Fill in the information on this page, and make sure your test booklet contains 11 pages. In addition, you should have a **multiple-choice answer sheet**, on which you should fill in your name, number, tutorial time, tutorial room, and tutor's name.

This test consists of 10 multiple choice questions, and 4 written-answer questions.
For the **multiple choice questions** you can do your rough work in the test booklet, but you must record your answer by circling the appropriate letter **on the answer sheet** with your pencil. Each correct answer is worth 4 marks; a question left blank, or an incorrect answer, or two answers for the same question is worth 0. For the **written-answer questions**, present your solutions in the space provided. The value of each written-answer question is indicated beside it.

ENCLOSE YOUR FINAL ANSWER IN A BOX AND WRITE IT IN INK.

TOTAL MARKS: 100

FAMILY NAME: ______________________

GIVEN NAME: ______________________

STUDENT NO: ______________________

SIGNATURE: ______________________

TUTORIAL TIME and ROOM: ______________________

REGCODE and TIMECODE: ______________________

T.A.'S NAME: ______________________

Regcode	Timecode	Room	**Regcode**	Timecode	Room
T0101A	M9A	SS1072	T0601A	R4A	RW142
T0101B	M9B	SS1073	T0601B	R4B	GB248
T0101C	M9C	SS1083	T0601C	R4C	AB107
T0201A	M3A	SS2106	T0601D	R4D	GB221
T0201B	M3B	MP134	T0701A	F2A	LM155
T0201C	M3C	RW143	T0701B	F2B	RW229
T0201D	M3D	UC328	T0701C	F2C	BA1240
T0301A	T3A	RW229	T0701D	F2D	MS4279
T0301B	T3B	WI524	T0801A	F3A	SS2135
T0301C	T3C	MP137	T0801B	F3B	SS2110
T0401A	W9A	SS1072	T0801C	F3C	WI 523
T0401B	W9B	SS1073	T5101A	M5A	SS1087
T0501A	W3A	SS1070	T5101B	M5B	MP134
T0501B	W3B	WI523	T5101C	M5C	MP137
			T5201A	M6A	LM162

FOR MARKER ONLY	
Multiple Choice	
B1	
B2	
B3	
B4	
TOTAL	

NAME: ____________________ STUDENT NO: ______________

PART A. Multiple Choice

1. *[4 marks]*

If $f(x) = \frac{x}{x^2+1}$ where $-2 \le x \le 4$ then f has

A. an absolute maximum at $x = 1$ and no absolute minimum

B. an absolute minimum at $x = -1$ and no absolute maximum

C. an absolute maximum at $x = 1$ and an absolute minimum at $x = -2$

D. an absolute maximum at $x = 1$ and an absolute minimum at $x = -1$

E. an absolute maximum at $x = 4$ and an absolute minimum at $x = -2$.

$f'(x) = \frac{(x^2+1) - x \cdot 2x}{(x^2+1)^2} = \frac{1-x^2}{(x^2+1)^2} = 0.$ Crit pts are at $x = \pm 1$ only.

Since f is cont. on $[-2,4]$ abs. max and abs. min. must be among

$f(-2) = -\frac{2}{5}$, $f(-1) = -\frac{1}{2}$, $f(1) = \frac{1}{2}$, $f(4) = \frac{4}{17}$

$\uparrow$ lowest $\quad \uparrow$ highest

$\boxed{\text{abs. min at } x=-1;\ \text{abs. max at } x=1}$ D

2. *[4 marks]*

If the total cost of producing q units is given by: $c = q^2 - 160q + 10,000$ then the value of q which minimizes the **average** cost is:

A. 80

B. 140

C. 100

D. 20

E. 1000

$\bar{c} = \frac{c}{q} = q - 160 + \frac{10,000}{q}$

$\frac{d\bar{c}}{dq} = 1 - \frac{10,000}{q^2} = 0$ when $\boxed{q = 100}$ C

$\frac{d\bar{c}}{dq}$: 0 — $-$ — 100 — $+$ $\rightarrow$ so $q = 100$ really is the abs. min on $(0, \infty)$.

NAME: ____________________ STUDENT NO: ____________

3. *[4 marks]*

Compute $f'(\sqrt{\ln 5})$ if

$$f(x) = \int_0^x t^2 e^{t^2}\,dt + \int_x^x \ln(2t)\,dt$$

Since $\int_x^x (\text{anything})\,dt = 0$,

$f'(x) = x^2 e^{x^2}$ by Fund. Thm.

$f'(\sqrt{\ln 5}) = (\ln 5)e^{\ln 5}$

$= \boxed{5\ln 5}$ B

A. 0

B. $5\ln 5$

C. $5\ln 5 + \ln(2\sqrt{\ln 5})$

D. $10\sqrt{\ln 5}(1+\ln 5)$

E. $10\sqrt{\ln 5}(1+\ln 5) + \frac{1}{2\sqrt{\ln 5}}$

4. *[4 marks]*

Compute the consumers' surplus if the demand equation is $p = \frac{4}{1+q}$ and the supply equation is $p = q+1$, where p is price per unit and q is the number of units of the product.

Supply = Demand

$\frac{4}{1+q} = q+1$

$(q+1)^2 = 4$

$q+1 = \pm 2$

$q = -1 \pm 2 = 1$ or -3

$q_0 = 1 \quad p_0 = 2$

$CS = \int_0^{q_0} [D(q) - p_0]\,dq$

$= \int_0^1 \left(\frac{4}{1+q} - 2\right)dq = \left[4\ln(1+q) - 2q\right]_0^1$

$= \boxed{4\ln 2 - 2}$ B

A. .5

B. $4(\ln 2) - 2$

C. $4(\ln 2) - 1.5$

D. $4\ln 2$

E. $4(\ln 2) + 0.5$

NAME: ______________________ STUDENT NO: ______________

5. *[4 marks]*

$$\int_1^2 \frac{x^2}{\sqrt{x^3+1}}dx =$$

Let $u = x^3+1$ $du = 3x^2 dx$

$\frac{1}{3}\int_2^9 \frac{du}{\sqrt{u}} = \frac{2}{3}\sqrt{u}\Big|_2^9$

$= \boxed{\frac{2}{3}(3-\sqrt{2})}$

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A. $\frac{2}{3}(3-\sqrt{2})$

B. $\frac{2}{3}(\sqrt{2}-1)$

C. $\frac{1}{3}\ln 2$

D. $\frac{1}{2}\left(\frac{1}{2\sqrt{2}} - \frac{1}{27}\right)$

E. $\frac{4}{3} - \frac{1}{\sqrt{2}}$

6. *[4 marks]*

If $f'(x) = e^{3x} + 2e^{-x}$ and $f(0) = \frac{1}{3}$, then $f(1) =$

A. $e^3 + \frac{2}{e}$

B. $\frac{e^3}{3} - \frac{2}{e} + 2$

C. $\frac{e^3}{3} - \frac{2}{e}$

D. $\frac{e^3}{3} - \frac{2}{e} - 2$

E. $e^3 + \frac{2}{e} + \frac{1}{3}$

$f(x) = \int (e^{3x} + 2e^{-x})dx$

$= \frac{e^{3x}}{3} - 2e^{-x} + C$

$\frac{1}{3} = f(0) = \frac{1}{3} - 2 + C$

so $C = 2$

$f(x) = \frac{e^{3x}}{3} - 2e^{-x} + 2$

$\boxed{f(1) = \frac{e^3}{3} - \frac{2}{e} + 2}$

B

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NAME: ____________________ STUDENT NO: ______________

7. *[4 marks]*

$\int_0^2 (1-x)e^x dx =$

Let $u = 1-x$, $du = -dx$; $dv = e^x dx$, $v = e^x$

$(1-x)e^x \Big|_0^2 + \int_0^2 e^x dx$

$= -e^2 - 1 + e^2 - 1 = \boxed{-2}$ A

A. -2

B. $e^2 - 1$

C. 0

D. $-1 - e^2$

E. $1 - e^2$

8. *[4 marks]*

At any time t years after the present, where $0 \le t \le 10$, cash flows into an account at the rate $1000e^{.05t}$ dollars per year. To the nearest $10, what is the present value of the cash flow if it ends in 10 years and the account always earns 3% compounded continuously?

A. $12,560

B. $11,070

C. $11,390

D. $10,930

E. $11,740

$P.V. = \int_0^{10} 1000e^{.05t} e^{-.03t} dt$

$= 1000 \int_0^{10} e^{.02t} dt$

$= \frac{1000}{.02} e^{.02t} \Big|_0^{10}$

$= 50,000(e^{.2} - 1)$

$\approx \boxed{\$11,070}$ B

NAME: ______________________ STUDENT NO: ______________

9. *[4 marks]*

$$\int_1^\infty \frac{dx}{\sqrt{x}} = \lim_{R\to\infty} \int_1^R x^{-\frac{1}{2}}\,dx$$

$$= \lim_{R\to\infty} 2\sqrt{x}\Big|_1^R$$

$$= \lim_{R\to\infty} 2(\sqrt{R}-1) \to \infty$$

so $\boxed{\text{diverges}}$ E

A. $= 2$

B. $= \frac{1}{2}$

C. $= 1$

D. $= \sqrt{2}$

E. diverges

10. *[4 marks]*

If $\dfrac{dy}{dx} = \dfrac{4x}{y}$ and $y = 2$ when $x = 1$, what is the value of y when $x = 3$?

A. 4

B. 8

C. 6

D. 10

E. 12

$$y\,dy = 4x\,dx$$

$$\frac{y^2}{2} = 2x^2 + C$$

$$y^2 = 4x^2 + K$$

$$4 = 4 + K \quad \text{so } K = 0$$

$$y^2 = 4x^2$$

$$y = \pm 2x \quad (\text{but } y = 2 \text{ when } x = 1)$$

so $y = 2x$

when $x = 3$ $\boxed{y = 6}$ C

NAME: ______________________ STUDENT NO: ______________

PART B. Written-Answer Questions

1. *[15 marks]*

Given the function $f(x) = \frac{4-4x^3}{8+x^3}$

[2] (a) Find the horizontal and vertical asymptotes of f (justify your answer in each case)

$\lim_{x \to \pm\infty} f(x) = -4$ either by knowledge of polynomial ratios or factoring out $\frac{x^3}{x^3}$ or L'Hôpital.

H.A. at $+\infty$ and $-\infty$ is $y = -4$

V.A. at $x = -2$

[3] (b) Given that $f'(x) = -\frac{108x^2}{(8+x^3)^2}$, find where f is increasing, decreasing, and all relative maximum and minimum points.

Interval	f'	f
$(-\infty, -2)$	$-$	dec
$(-2, 0)$	$-$	dec
$(0, \infty)$	$-$	dec

since $f' = -[\text{square}]$

There are no local extrema

[4] (c) Given that $f''(x) = \frac{432x(x^3-4)}{(8+x^3)^3}$, find where f is concave upward and downward and all inflection points.

Int.	f''	f
$(-\infty, -2)$	$-$	concave down
$(-2, 0)$	$+$	conc. up
$(0, 4^{\frac{1}{3}})$	$-$	conc. down
$(4^{\frac{1}{3}}, \infty)$	$+$	conc. up

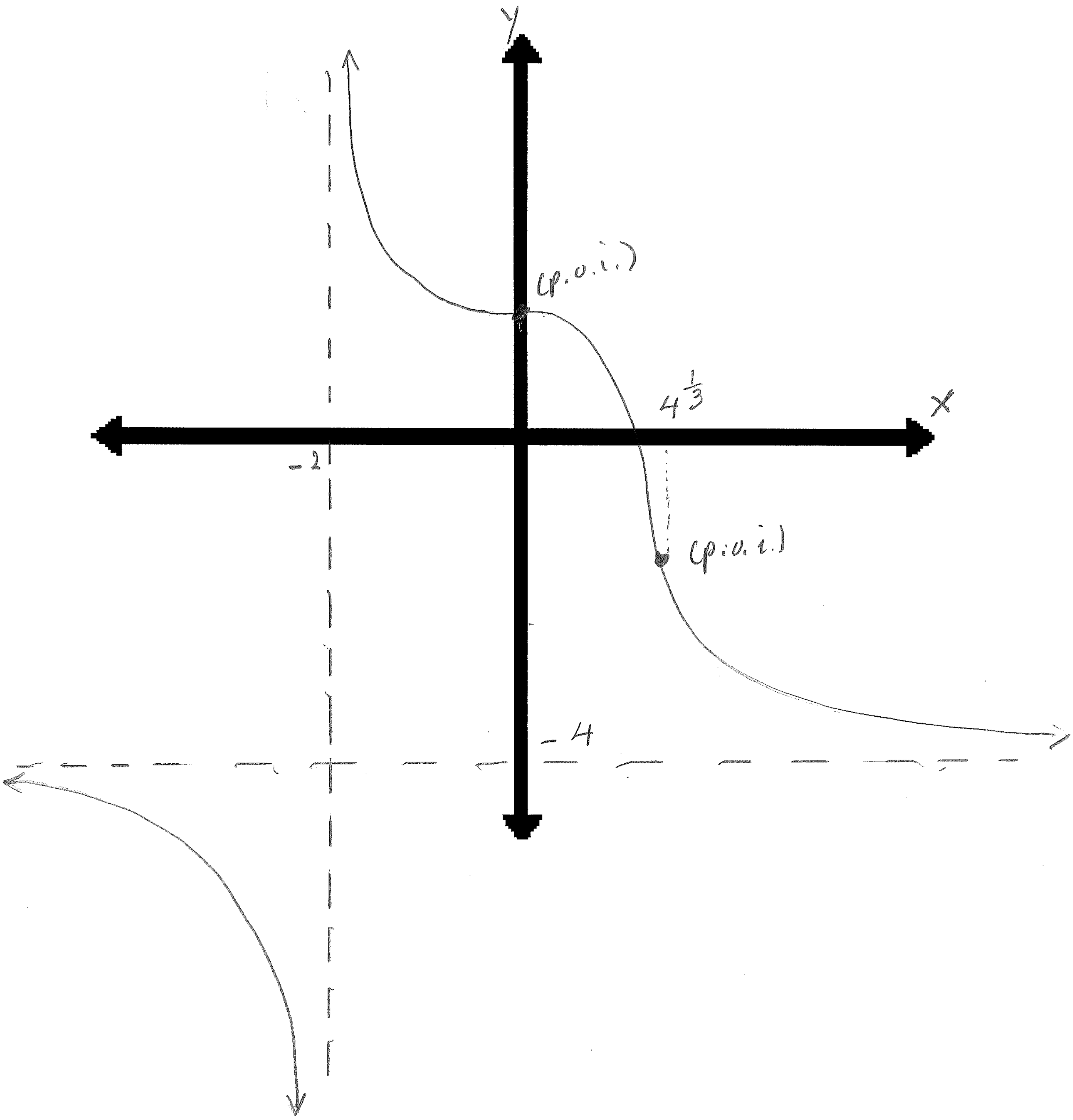
$x = 0$ and $x = 4^{\frac{1}{3}}$ } are points of inflection

Concavity changes at $x = -2$ but this is not a point on the curve so not a point of inflection.

[6] (d) Graph f clearly on the axes on the following page.

A

NAME: ______________________ STUDENT NO: ______________



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NAME: ______________________ STUDENT NO: ______________

2. *[14 marks]*

A retailer bought 1000 cases of an imported wine for \$75. As the wine ages, its value increases, but once it has reached its prime, the value of the wine drops. Suppose that the price of a case of wine in t years is given by:

$$p = 53t - 10t^2 + 100.$$

If the retailer pays \$3/case/year to store the wine, when should he sell the wine in order to maximize his profits? Verify that this is the maximum profit and that it is more than he will get if he sells the wine right away.

Note that the storage cost is prorated over parts of the year. For example, if the retailer stores the wine for $1\frac{1}{3}$ years, he would pay \$4/case total.

$R = 1000p = 1000(53t - 10t^2 + 100)$ $\quad \Pi = R - C$

$C = 75{,}000 + 3000t$

Profit $= \Pi = 1000[-10t^2 + 50t + 25]$

$\frac{d\Pi}{dt} = 1000(-20t + 50)$

$\frac{d\Pi}{dt} = 0$ when $\boxed{t = \frac{5}{2} \quad (2.5 \text{ yrs})}$

Verification:

Method A: $\frac{d^2\Pi}{dt^2} = -20{,}000 < 0$ for all values of t

so the local max is indeed a global max.

Method B: Π is a parabola with ~~vertex~~ pointing up

so its max is at ~~the vertex~~ where $\frac{d\Pi}{dt} = 0$

Method C: $\frac{d\Pi}{dt} = -20{,}000(t - \frac{5}{2})$

$\frac{d\Pi}{dt} > 0$ when t in $(0, \frac{5}{2})$

and $\frac{d\Pi}{dt} < 0$ when t in $(\frac{5}{2}, \infty)$

so at $t = \frac{5}{2}$, Π is as high as it can be.

Method D: $\Pi \to -\infty$ as $t \to \infty$

and $\Pi(0) = 25{,}000$

$\Pi(2.5) = 62{,}500$ which is more

NAME: ______________________ STUDENT NO: ______________

3. *[15 marks]*

Compute the total area between $y = -x^2 - 5x + 6$ and $y = x^2 + x - 2$ from $x = -4$ to $x = 3$.

Intersect at $\quad -x^2 - 5x + 6 = x^2 + x - 2$

$-2x^2 - 6x + 8 = 0$

$-2(x^2 + 3x - 4) = 0$

$-2(x+4)(x-1) = 0$

$x = -4$ and $x = 1$

on $[-4, 1]$ $\quad x^2 - 5x + 6$ is above $x^2 + x - 2$ (try $x = 0$)

on $[1, 3]$ $\quad x^2 + 5x - 2$ " " $x^2 - 5x + 6$ (try $x = 2$)

$$\text{Total area} = \int_{-4}^{1} [(-x^2 - 5x + 6) - (x^2 + x - 2)]\,dx + \int_{1}^{3} [(x^2 + x - 2) - (-x^2 - 5x + 6)]\,dx$$

$$= \int_{-4}^{1} [-2x^2 - 6x + 8]\,dx + \int_{1}^{3} [2x^2 + 6x - 8]\,dx$$

$$= \left[-\frac{2x^3}{3} - 3x^2 + 8x\right]_{-4}^{1} + \left[\frac{2x^3}{3} + 3x^2 - 8x\right]_{1}^{3}$$

$$= \left[\left(-\frac{2}{3} - 3 + 8\right) - \left(\frac{128}{3} - 48 - 32\right)\right] + \left[(18 + 27 - 24) - \left(\frac{2}{3} + 3 - 8\right)\right]$$

$$= \left[-\frac{130}{3} + 85\right] + \left[26 - \frac{2}{3}\right]$$

$$= 111 - \frac{132}{3} = 111 - 44 = \boxed{67}$$

NAME: __________________ STUDENT NO: ______________

4. *[16 marks]*

[8] (a) Find the average value, on the interval $[1, e]$, of $f(x) = x \ln x$.

$$\bar{f} = \frac{1}{e-1}\int_1^e x \ln x \, dx \qquad u = \ln x \quad dv = x\,dx \qquad du = \frac{dx}{x} \quad v = \frac{x^2}{2}$$

$$= \frac{1}{e-1}\left[\frac{x^2}{2}\ln x \Big|_1^e - \int_1^e \frac{x}{2}\,dx \right]$$

$$= \frac{1}{e-1}\left[\frac{e^2}{2} - \frac{x^2}{4}\Big|_1^e \right] = \frac{1}{e-1}\left[\frac{e^2}{2} - \frac{e^2}{4} + \frac{1}{4} \right] = \frac{1}{e-1}\left[\frac{e^2}{4} + \frac{1}{4} \right]$$

$$= \boxed{\frac{e^2+1}{4(e-1)}}$$

[8] (b) Find $\int \frac{dx}{x^2(x+1)}$.

$$\frac{A}{x} + \frac{B}{x^2} + \frac{C}{x+1} \qquad A x(x+1) + B(x+1) + Cx^2 = 1$$

$x = 0 \Rightarrow B = 1$

$x = -1 \Rightarrow C = 1$

$x = 1 \Rightarrow 2A + 2B + C = 1$

$2A + 2 + 1 = 1$

$A = -1$

so
$$\int \frac{dx}{x^2(x+1)} = \int \left[-\frac{1}{x} + \frac{1}{x^2} + \frac{1}{x+1} \right] dx$$

$$= \boxed{-\ln|x| - \frac{1}{x} + \ln|x+1| + C}$$

or $\boxed{\left(\ln\left|\frac{x+1}{x}\right|\right) - \frac{1}{x} + C}$