Solved

A

Department of Mathematics
University of Toronto

Tuesday, March 1, 2011, 6:10 - 8:00 PM
MAT 133Y TERM TEST #3

Calculus and Linear Algebra for Commerce
Duration: 1 hour 50 minutes

Aids Allowed: A non-graphing calculator, with empty memory, to be supplied by student.

Instructions: Fill in the information on this page, and make sure your test booklet contains 10 pages. In addition, you should have a **multiple-choice answer sheet**, on which you should fill in your name, number, tutorial time, tutorial room, and tutor's name.

This test consists of 10 multiple choice questions, and 4 written-answer questions.
For the **multiple choice questions** you can do your rough work in the test booklet, but you must record your answer by circling the appropriate letter **on the answer sheet** with your pencil. Each correct answer is worth 4 marks; a question left blank, or an incorrect answer, or two answers for the same question is worth 0. For the **written-answer questions**, present your solutions in the space provided. The value of each written-answer question is indicated beside it.

ENCLOSE YOUR FINAL ANSWER IN A BOX AND WRITE IT IN INK.

TOTAL MARKS: 100

FAMILY NAME: ______________________

GIVEN NAME: ______________________

STUDENT NO: ______________________

SIGNATURE: ______________________

TUTORIAL TIME and ROOM: ______________________

REGCODE and TIMECODE: ______________________

T.A.'S NAME: ______________________

Regcode	Timecode	Room	Regcode	Timecode	Room
T0101A	M9A	SS1074	T0601A	R4A	SS2106
T0101B	M9B	SS1084	T0601B	R4B	LM 123
T0201A	M3A	LM 155	T0601C	R4C	SS1069
T0201B	M3B	RW 143	T0601D	R4D	LM 158
T0201C	M3C	SS2127	T0701A	F2A	LM 123
T0201D	M3D	SS1074	T0701B	F2B	LM 157
T0301A	T3A	SS1070	T0701C	F2C	AP 120
T0301B	T3B	SS1084	T0701D	F2D	RW 229
T0301C	T3C	WI524	T0801A	F3A	MP 134
T0401A	W9A	SS1074	T0801B	F3B	SS1074
T0401B	W9B	SS1084	T0801C	F3C	WI 524
T0401C	W9C	SS1088	T5101A	M5A	MP 134
T0501A	W3A	LM 123	T5101B	M5B	SS1073
T0501B	W3B	LM 157	T5101C	M5C	LM 155
			T5201A	M6A	LM 162

FOR MARKER ONLY	
Multiple Choice	
B1	
B2	
B3	
B4	
TOTAL	

NAME: ____________________ STUDENT NO: ________________

PART A. Multiple Choice

1. *[4 marks]*

Given the demand function

$$p = 1000qe^{-q/100}$$

unit elasticity of demand occurs when $q =$

A. 50

B. 75

C. 100

D. 150

E. 200

$$\eta = \frac{\frac{dq}{q}}{\frac{dp}{p}} = \frac{p}{q\frac{dp}{dq}} = \frac{\cancel{1000}\cancel{q}e^{-\frac{q}{100}}}{\cancel{q}\times\cancel{1000}\left[e^{-\frac{q}{100}} - \frac{q}{100}e^{-\frac{q}{100}}\right]}$$

$$= \frac{1}{1-\frac{q}{100}}$$

$\eta = 1$ when $q = 0$

but $\eta = -1$ when $1 - \frac{q}{100} = -1$

$\boxed{q = 200}$ E

2. *[4 marks]*

Using Newton's method to find a solution to the equation

$$x^4 - 2x^3 + x^2 - 3 = 0$$

and starting with $x_0 = 2$, x_1 is closest to

A. 1.90785

B. 1.90794

C. 1.91667

D. 1.91985

E. 1.92124

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$$

$$= x_0 - \frac{x_0^4 - 2x_0^3 + x_0^2 - 3}{4x_0^3 - 6x_0^2 + 2x_0}$$

If $x_0 = 2$,

$$x_1 = 2 - \frac{16 - 16 + 4 - 3}{32 - 24 + 4} = 2 - \frac{1}{12}$$

$\boxed{x_1 \approx 1.91667}$ C

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3. *[4 marks]*

If $y = (x^2 + x + 1)^x$ then $y' =$

$$\ln y = x \ln(x^2+x+1)$$

$$\frac{1}{y} y' = \ln(x^2+x+1) + \frac{x(2x+1)}{x^2+x+1}$$

$$y' = y\left[\ln(x^2+x+1) + \frac{(2x^2+x)}{x^2+x+1}\right]$$

C

A. $x(x^2 + x + 1)^{x-1}(2x + 1)$

B. $\ln(x^2 + x + 1) + \dfrac{2x^2 + x}{x^2 + x + 1}$

C. $(x^2 + x + 1)^x \left[\ln(x^2 + x + 1) + \dfrac{2x^2 + x}{x^2 + x + 1}\right]$

D. $\dfrac{(x^2 + x + 1)^x}{2x + 1}$

E. $(x^2 + x + 1)^x \ln(x^2 + x + 1)$

4. *[4 marks]*

The slope of the tangent to the curve

$$1 + x^2\sqrt{y + 1} = y^2\sqrt{x + 1}$$

at the point $(0, 1)$ is

$$2x\sqrt{y+1} + \frac{x^2}{2\sqrt{y+1}} y' = 2yy'\sqrt{x+1} + \frac{y^2}{2\sqrt{x+1}}$$

When $x=0$ and $y=1$

$$0 = 2y' + \frac{1}{2}$$

$$\boxed{y' = -\frac{1}{4}}$$

A

A. $-\dfrac{1}{4}$

B. $\dfrac{1}{2}$

C. 0

D. 1

E. $-\dfrac{1}{2}$

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5. *[4 marks]*

Let $y = x^x$. Then $y'' =$

A. $x(x-1)x^{x-2}$

B. x^x

C. $\ln(x^x)$

D. $x^x(1+\ln x)^2 + \ln(x^x)$

E. $x^x(1+\ln x)^2 + x^{x-1}$

$\ln y = x \ln x$

$\frac{1}{y} y' = \ln x + 1$

$y' = y(1+\ln x)$

$y'' = y'(1+\ln x) + y \cdot \frac{1}{x}$

$= y(1+\ln x)^2 + \frac{y}{x}$

$= \boxed{x^x(1+\ln x)^2 + x^{x-1}}$ E

6. *[4 marks]*

The function $f(x) = 2x\sqrt{x+3}$ on the interval $[-3, 1]$ has:

A. An absolute minimum at $x = -2$ and an absolute maximum at $x = -3$.

B. An absolute minimum at $x = -3$ and an absolute maximum at $x = 1$.

C. An absolute minimum at $x = -2$ and no absolute maximum.

D. An absolute minimum at $x = -2$ and an absolute maximum at $x = 1$.

E. An absolute minimum at $x = -3$ and absolute maxima at $x = -2$ and $x = 1$.

$f'(x) = 2\sqrt{x+3} + \frac{2x}{2\sqrt{x+3}}$

$= 0$ when $2(x+3) + x = 0$ i.e. when $x = -2$

f is cont. on $[-3,1]$ so must have an absolute max and min on $[-3,1]$.

$\left.\begin{array}{l} f(-3) = 0 \\ f(-2) = -4 \\ f(1) = 4 \end{array}\right\}$ f has an absolute max at $x = 1$
" " " min at $x = -2$

D

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7. *[4 marks]*

If $c = q^2 - 2q + 100$ is the total cost of producing q units then the average cost is minimized when $q =$

$\bar{c} = \frac{c}{q} = q - 2 + \frac{100}{q}$

$\frac{d\bar{c}}{dq} = 1 - \frac{100}{q^2} = \frac{q^2 - 100}{q^2} = 0$ when $q = \pm 10$

no derivative at $q = 0$.

$+ \quad - \quad - \quad +$

$-10 \quad 0 \quad 10$

Actually, q must be positive and $\bar{c}$ decreases all the way from 0 to 10 and increases afterwards.

Min is at $\boxed{q = 10}$ B

A. 1

B. 10

C. $\sqrt{10}$

D. ± 10

E. 100

8. *[4 marks]*

If $f(x) = \int_0^x \sqrt{t^2+9}\,dt$ then $f'(4) =$

$f'(x) = \sqrt{x^2+9}$ (Fundamental Theorem)

$f'(4) = \sqrt{25} = \boxed{5}$

E

A. $\frac{2}{5}$

B. 4

C. $\frac{4}{5}$

D. 3

E. 5

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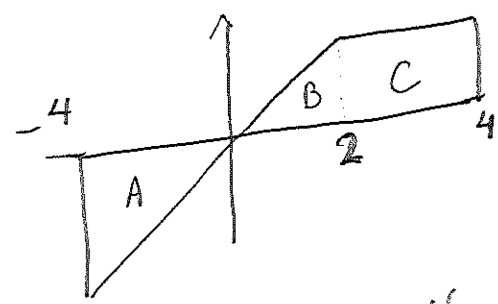
9. *[4 marks]*

If $f(x) = \begin{cases} x & \text{if } -4 \le x \le 2 \\ 2 & \text{if } 2 \le x \le 4 \end{cases}$

then $\cancel{\int_{4}^{-4} f(x)dx} =$ $\int_{-4}^{4} f(x)dx =$



A. 0

B. −2

C. 4

D. 14

E. −8

Easiest way is geometrically

$\int_{-4}^{0} f(x) = -\frac{4 \times 4}{2}$ $\int_{0}^{2} f(x) = \frac{2 \times 2}{2}$ $\int_{2}^{4} f = 2 \times 2$

Adding: $-8 + 2 + 4 = \boxed{-2}$ B

Alternatively: $\int_{-4}^{4} f(x)dx = \int_{-4}^{2} x\,dx + \int_{2}^{4} 2\,dx$

$= \frac{x^2}{2}\Big|_{-4}^{2} + 2x\Big|_{2}^{4} = \left(\frac{4}{2} - \frac{16}{2}\right) + (8-4) = -2$

as before.

10. *[4 marks]*

If a is a positive constant, then

$$\lim_{x \to 0} (2 - a^x)^{\frac{1}{x}}$$

A. is not defined

B. $= a$

C. $= e^a$

D. $= -\ln a$

E. $= \frac{1}{a}$

$y = (2 - a^x)^{\frac{1}{x}}$

$\ln y = \frac{\ln(2 - a^x)}{x}$ $\frac{\ln(2-1)}{0} = \frac{0}{0}$ at $x=0$

$\lim_{x \to 0} \ln y = \lim_{x \to 0} -\frac{a^x \ln a}{1} = -\ln a$

so $y \to e^{-\ln a} = \frac{1}{e^{\ln a}} = \boxed{\frac{1}{a}}$

E

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PART B. Written-Answer Questions

1. *[15 marks]*

Given $f(x) = \dfrac{x}{(x+2)^2}$, $f'(x) = \dfrac{2-x}{(2+x)^3}$, $f''(x) = \dfrac{2(x-4)}{(x+2)^4}$,

[1] (a) Find all intercepts of $y = f(x)$.

The only intercept, x and y, is $\boxed{(0,0).}$

[2] (b) Find all vertical and horizontal asymptotes of $y = f(x)$.

$\boxed{\text{V.A.}: x = -2}$

H.A. $\lim_{x \to \pm\infty} \dfrac{x}{(x+2)^2} = 0$ so $\boxed{y = 0 \text{ at } \pm\infty}$

[3] (c) Find where f is increasing, decreasing, and all relative extrema.

	f'	f
$(-\infty,-2)$	$-$	dec.
$(-2, 2)$	$+$	inc
$(2,\infty)$	$-$	dec

$x=-2$ is an H.A. so no local extremum.

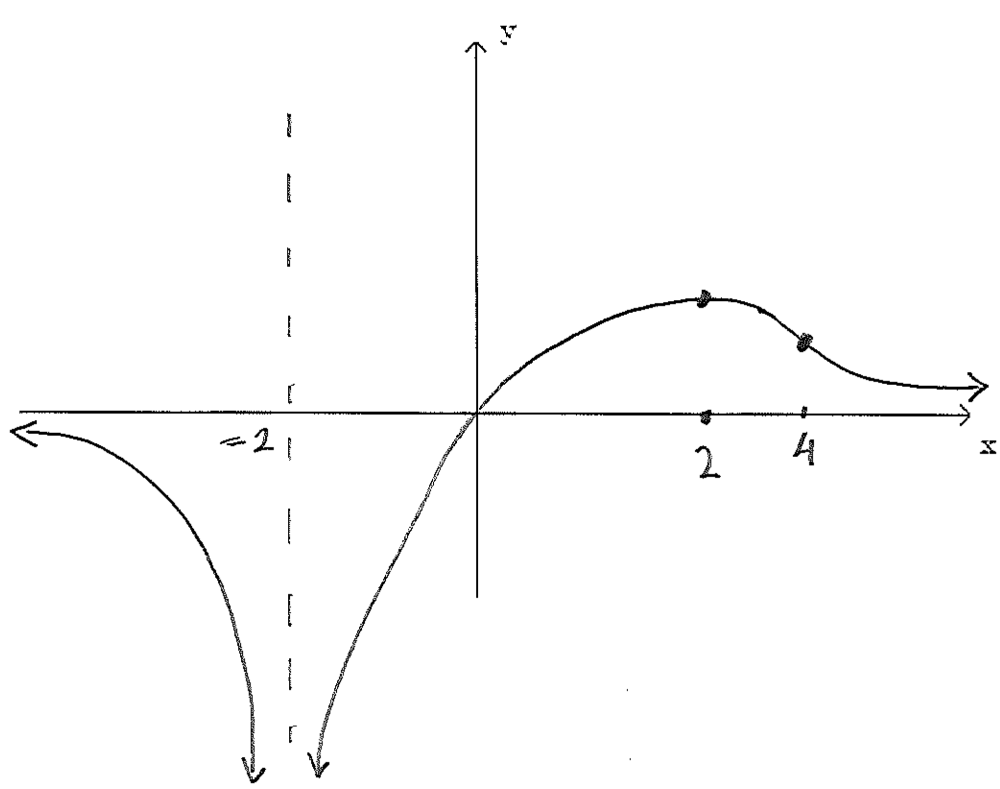
$\wedge$ $\boxed{x = 2 \text{ is a local max}}$ (actually global).

[3] (d) Find where the graph of $y = f(x)$ is concave up, concave down, and all inflection points.

	f''	f
$(-\infty,-2)$	$-$	conc. down
$(-2, 4)$	$-$	conc. down
$(4,\infty)$	$+$	conc. up

$\boxed{x=4 \text{ is the only inflection point}}$

[6] (e) Draw a clearly labelled graph of $y = f(x)$ on the axes below.



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2. *[14 marks]*

A car dealership sells 200 cars/month when they charge an average price of \$24,000. For each \$1000 that they take off the price, they will sell 25 more cars/month. If the cost to the dealership of each car is \$12000 then what price should they charge in order to maximize their profits ? (Make sure to justify your answer.) What will the maximum profit/month be?

Let x be the number of \$1000 reductions.

$p = 24{,}000 - 1000x$ price

$q = 200 + 25x$ quantity

$C = 12{,}000\,q$ cost

profit: $\Pi = R - C = pq - C$

$\Pi = (24{,}000 - 1000x)(200 + 25x) - 12{,}000\,(200 + 25x)$

$\Pi = (12{,}000 - 1000x)(200 + 25x)$

$\Pi = 25{,}000(12 - x)(8 + x)$

$\frac{d\Pi}{dx} = 25{,}000\left[-(8+x) + (12-x)\right] = 25{,}000(4 - 2x)$

$\frac{d\Pi}{dx} = 0$ when $\boxed{x = 2}$ $\boxed{p = \$22{,}000}$ $\boxed{\Pi = \$2{,}500{,}000}$

$(-\infty, 2)$: $\frac{d\Pi}{dx} > 0$ Π increasing

$(2, \infty)$: $\frac{d\Pi}{dx} < 0$ Π decreasing

$\boxed{\text{max at } x = 2}$

OR: $\frac{d^2\Pi}{dx^2} = -50{,}000 < 0$ for all values of x

max at $x = 2$

note; Just saying $\frac{d^2\Pi}{dx^2} < 0$ does not justify the answer.

OR: Π is a quadratic with coefficient of x^2 negative.

picture: $\cap$. So $\frac{d\Pi}{dx} = 0$ only at max

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3. *[16 marks]*

[8] (a) Find $\int_0^2 \frac{e^{3x}}{1+e^{3x}}dx$

Let $u = 1+e^{3x}$ $\quad du = 3e^{3x}\,dx$

$$\int_0^2 = \frac{1}{3}\int_2^{1+e^6} \frac{du}{u} = \frac{1}{3}\ln|u|\Big|_2^{1+e^6} = \boxed{\frac{1}{3}\left[\ln(1+e^6) - \ln 2\right]}$$

≈ 1.77

or $\int \frac{e^{3x}}{1+e^{3x}}dx$ $\quad u = 1+e^{3x}$ $\quad du = 3e^{3x}dx$

$$= \frac{1}{3}\int \frac{du}{u} = \frac{1}{3}\ln|u| = \frac{1}{3}\ln(1+e^{3x}).$$

$$\int_0^2 = \frac{1}{3}\left[\ln(1+e^{6x}) - \ln 2\right]$$ as before.

[8] (b) Find a function $f(x)$ such that

$$f'(x) = \frac{16x}{\sqrt{2x^2+1}} \quad \text{and } f(2) = 50$$

$f(x) = \int \frac{16x}{\sqrt{2x^2+1}}dx$ $\quad u = 2x^2+1$, $\quad du = 4x\,dx$

$= 4\int \frac{du}{\sqrt{u}}$

$= 4\sqrt{u}\cdot 2 + C$

$= 8\sqrt{2x^2+1} + C$

$50 = 8\sqrt{9} + C$ $\quad$ so $\boxed{C = 26}$

$$\boxed{f(x) = 26 + 8\sqrt{2x^2+1}}$$

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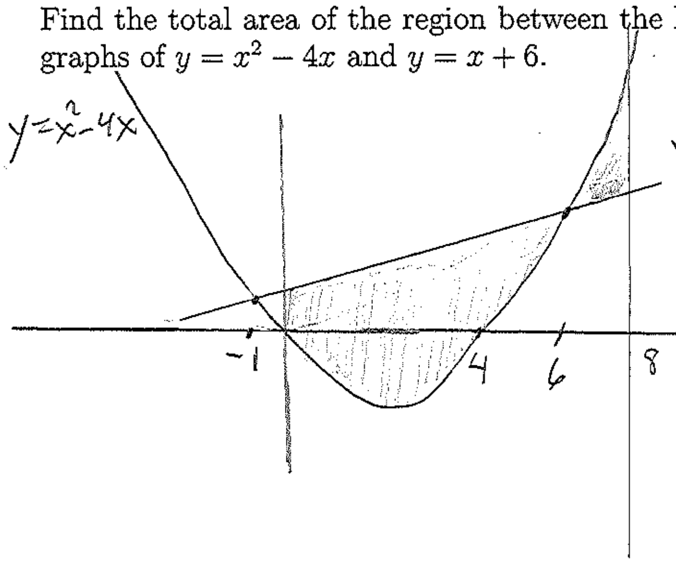
4. *[15 marks]*

Find the total area of the region between the lines $x = 0$ and $x = 8$ which lies between the graphs of $y = x^2 - 4x$ and $y = x + 6$.



Intersection:

$$x^2-4x = x+6$$
$$x^2-5x-6=0$$
$$(x-6)(x+1)=0$$
$$x=6,\ x=-1$$

The area asked for is the shaded region.

$$\int_0^6 \left[(x+6)-(x^2-4x)\right]dx + \int_6^8 \left[(x^2-4x)-(x+6)\right]dx$$

$$= \int_0^6 (-x^2+5x+6)\,dx + \int_6^8 (x^2-5x-6)\,dx$$

$$= \left(-\frac{x^3}{3}+\frac{5x^2}{2}+6x\right)_0^6 + \left(\frac{x^3}{3}-\frac{5x^2}{2}-6x\right)_6^8$$

$$= \left[\left(-\frac{216}{3}+\frac{180}{2}+36\right)-0\right] + \left[\left(\frac{512}{3}-\frac{320}{2}-48\right)-\left(\frac{216}{3}-\frac{180}{2}-36\right)\right]$$

$$= (-72+90+36) + (170\tfrac{2}{3}-160-48)-(72-90-36)$$

$$= 54+\tfrac{2}{3}-38+54 = \boxed{70\tfrac{2}{3}} \quad \text{or } \frac{212}{3} \text{ or } \approx 70.67$$