Solved

Department of Mathematics
University of Toronto

WEDNESDAY, March 5, 2008 6:10-8:00 PM
MAT 133Y TERM TEST #3

Calculus and Linear Algebra for Commerce
Duration: 1 hour 50 minutes

Aids Allowed: A non-graphing calculator, with empty memory, to be supplied by student.

Instructions: Fill in the information on this page, and make sure your test booklet contains 11 pages. In addition, you should have a **multiple-choice answer sheet**, on which you should fill in your name, number, tutorial time, tutorial room, and tutor's name.

This test consists of 10 multiple choice questions, and 4 written-answer questions.
For the **multiple choice questions** you can do your rough work in the test booklet, but you must record your answer by circling the appropriate letter **on the answer sheet** with your pencil. Each correct answer is worth 4 marks; a question left blank, or an incorrect answer, or two answers for the same question is worth 0. For the **written-answer questions**, present your solutions in the space provided. The value of each written-answer question is indicated beside it. ENCLOSE YOUR FINAL ANSWER IN A BOX AND WRITE IT IN INK.

TOTAL MARKS: 100

FAMILY NAME: ______________________

GIVEN NAME: ______________________

STUDENT NO: ______________________

SIGNATURE: ______________________

TUTORIAL TIME and ROOM: ______________________

REGCODE and TIMECODE: ______________________

T.A.'S NAME: ______________________

Regcode	Timecode	Room	**Regcode**	Timecode	Room
T0101A	M9A	SS1084	T0501D	W3D	SS2108
T0101B	M9B	SS1086	T0601A	R4A	LM 157
T0101C	M9C	SS1087	T0601B	R4B	SS1083
T0201A	M3A	SS2108	T0701A	F2A	SS1086
T0201B	M3B	RW 143	T0701B	F2B	SS2106
T0201C	M3C	ES142	T0701C	F2C	SS2108
T0201D	M3D	RW 142	T0801A	F3A	MP 134
T0301A	T3A	SS1084	T0801B	F3B	MP 118
T0301B	T3B	SS2108	T5101A	M5A	MP 118
T0401A	W9A	SS1084	T5101B	M5B	WI 523
T0401B	W9B	SS1073	T5201A	M6A	LM 162
T0501A	W3A	SS1086	T5201B	M6B	SS2106
T0501B	W3B	SS1083			
T0501C	W3C	SS2106			

FOR MARKER ONLY	
Multiple Choice	
B1	
B2	
B3	
B4	
TOTAL	

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PART A. Multiple Choice

1. *[4 marks]*

   If b is a constant, $\lim_{x \to 0} \frac{1 + bx - e^{bx}}{x^2}$

   $\frac{0}{0}$ L'Hôpital

   A. is undefined

   B. is 0

   C. is $1 - b$

   D. is $-\frac{b}{2}$

   E. is $-\frac{b^2}{2}$

   $= \lim_{x \to 0} \frac{b - be^{bx}}{2x}$ still $\frac{0}{0}$

   $= \lim_{x \to 0} \frac{-b^2 e^{bx}}{2} = \boxed{\frac{-b^2}{2}}$ E

2. *[4 marks]*

   $\lim_{x \to \infty} x^{\frac{1}{x}}$ $\quad \infty^0$

   A. is undefined

   B. is 1

   C. is 0

   D. is e

   E. is $\frac{1}{e}$

   Let $y = x^{\frac{1}{x}}$

   $\ln y = \frac{\ln x}{x}$ $\quad \frac{\infty}{\infty}$

   $\lim_{x \to \infty} \ln y = \lim_{x \to \infty} \frac{\frac{1}{x}}{1} = 0$

   $\ln y \to 0$

   $e^{\ln y} \to e^0 = \boxed{1}$ B

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3. *[4 marks]*

The function $f(x) = \dfrac{x}{(x-1)^2}$ on the interval $[-2, 2]$ has

A. an absolute minimum at $x = 2$ and an absolute maximum at $x = -2$.

B. an absolute minimum at $x = -2$ and no absolute maximum

C. an absolute minimum at $x = -1$ and no absolute maximum

D. an absolute minimum at $x = -1$ and an absolute maximum at $x = 2$

E. no absolute minimum or maximum

$$f'(x) = \frac{(x-1)^2 - x\cdot 2(x-1)}{(x-1)^4} = -\frac{(x+1)}{(x-1)^3}$$

V.A. at $x = 1$.
critical pt at $x = -1$

dec inc dec
$-2 \quad -1 \quad 1 \quad 2$

Since $\lim_{x\to 1} f(x) = +\infty$ (on both sides)
there is $\boxed{\text{no absolute max}}$

$f(-2) = -\frac{2}{9}$ $\quad f(-1) = -\frac{1}{4}$ $\quad f(2) = 2$

$-.25 < -.22$

f has $\boxed{\text{an absolute minimum at } x = -1}$

C

4. *[4 marks]*

If $f(x) = \dfrac{1}{\sqrt{g(x)}}$, then if g is always positive, increasing and concave downward, wherever g is defined, then

A. f is always decreasing and concave upward

B. f is always increasing and concave upward

C. f is always decreasing and concave downward

D. f is always increasing and concave downward

E. f is always decreasing but may be sometimes concave upward and sometimes concave downward

$f(x) = [g(x)]^{-\frac{1}{2}}$

$f'(x) = -\frac{1}{2}[g(x)]^{-\frac{3}{2}} g'(x) < 0$ since $g \uparrow \Rightarrow g' > 0$.

$\boxed{f \text{ is decreasing}}$

$f''(x) = \frac{3}{4}[g(x)]^{-\frac{5}{2}}[g'(x)]^2 - \frac{1}{2}[g(x)]^{-\frac{3}{2}} g''(x)$

but $g''(x) < 0$, so $f''(x) > 0$ always

$\boxed{f \text{ is concave upward}}$

A

5. *[4 marks]*
If the total cost of producing q units of a product is given by

$$c = 250 + 6q + 0.1q^2$$

then the **average** cost will be a minimum when q is

A. 30

B. $5\sqrt{10}$

C. -30

D. 50

E. 5

$\bar{c} = \frac{250}{q} + 6 + .1q$

$\frac{d\bar{c}}{dq} = -\frac{250}{q^2} + .1$

$\frac{d\bar{c}}{dq} = 0$ when $q^2 = 2500$

$\boxed{q = 50}$ D

$q < 50$ $\bar{c}' < 0$ $q > 50$ $\bar{c}' > 0$

So this really is an absolute min..

Or $\frac{d^2\bar{c}}{dq^2} = \frac{500}{q^3} > 0$ for <u>every</u> q in the domain.

6. *[4 marks]*
If $f''(x) = 24x$, $f(0) = 4$, and $f'(0) = 6$, then $f(1) =$

A. 16

B. 17

C. 14

D. 15

E. 18

$f'(x) = 12x^2 + C$

$6 = 12 \cdot 0 + C$

so $f'(x) = 12x^2 + 6$

$f(x) = 4x^3 + 6x + K$

$4 = 0 + 0 + K$

$f(x) = 4x^3 + 6x + 4$

$\boxed{f(1) = 14}$ C

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7. *[4 marks]*
If $f(x) = \int_0^x \frac{dt}{1+e^t}$, then $f'(\ln 2) =$

A. $\frac{1}{3}$

B. $\ln\left(\frac{2}{3}\right)$

C. $-\frac{2}{9}$

D. $-\ln 3$

E. $\ln\left(\frac{4}{3}\right)$

$f'(x) = \frac{1}{1+e^x}$

$f'(\ln 2) = \frac{1}{1+e^{\ln 2}} = \frac{1}{1+2}$

$= \boxed{\frac{1}{3}}$ A

8. *[4 marks]*
The area of the region bounded by the x-axis, the line $x = 2$, and the curve $y = x^{1/3}$ is

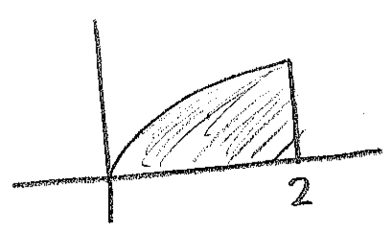
A. $\frac{8\sqrt[3]{2}}{3}$

B. $\frac{3\sqrt[3]{2}}{2}$

C. $1 - \sqrt[3]{2}$

D. 2

E. $8\sqrt[3]{2} - 1$



$\text{Area} = \int_0^2 x^{\frac{1}{3}}\,dx$

$= \frac{3}{4}x^{\frac{4}{3}}\Big|_0^2$

$= \frac{3}{4}2^{\frac{4}{3}} = \boxed{\frac{3}{2}\cdot 2^{\frac{1}{3}}}$ B

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9. *[4 marks]*

A manufacturer's marginal cost function is

$$\frac{dc}{dq} = \frac{500}{\sqrt{2q+40}}$$

where c is in dollars.

The cost to increase production from 12 to 30 units is

A. \$2000

B. \$1000

C. \$800

D. \$500

E. \$100

$$c(30) - c(12) = \int_{12}^{30} \frac{dc}{dq}\,dq$$

$$= \int_{12}^{30} \frac{500}{\sqrt{2q+40}}\,dq = \frac{2 \cdot 500\sqrt{2q+40}}{2}\Big|_{12}^{30}$$

$$= 500\left(\sqrt{100} - \sqrt{64}\right)$$

$$= \boxed{1000} \quad \text{B}$$

10. *[4 marks]*

The rational function $\dfrac{3x^2+1}{(x+1)(x+2)^2}$ is expressed as a sum $\dfrac{a}{x+1} + \dfrac{b}{x+2} + \dfrac{c}{(x+2)^2}$ where a, b, c are constants. Then $c =$

A. -4

B. 2

C. -13

D. -11

E. 0

$$a(x+2)^2 + b(x+1)(x+2) + c(x+1) = 3x^2+1$$

If $x = -2$ $\qquad -c = 13$

$$\boxed{c = -13} \quad \text{C}$$

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PART B. Written-Answer Questions

1. *[18 marks]*

Given: $y = xe^{-x}$

(i) Find the following:

[This space for rough work.]

$y' = e^{-x} - xe^{-x}$

$y'' = -e^{-x} - (1-x)e^{-x}$

$$\boxed{\begin{aligned} y' &= (1-x)e^{-x} \\ y'' &= (x-2)e^{-x} \end{aligned}}$$

[5] (a) where y is increasing, decreasing, and all relative extrema if any

crit pt at $x = 1$ only

	y'	y
$(-\infty, 1)$	+	inc
$(1, \infty)$	−	dec

$\wedge$

$\boxed{x = 1 \text{ is a local (and absolute) max}}$

[5] (b) where the graph is concave upward and downward and all inflection points if any

	y''	
$(-\infty, 2)$	−	conc. down
$(2, \infty)$	+	conc. up

$\boxed{x = 2 \text{ is a point of inflection}}$

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[2] (c) the horizontal and vertical asymptotes if any (justify your answer)

There is $\boxed{\text{no V.A}}$ since the function is cont. every where,

$\lim_{x \to -\infty} xe^{-x} = -\infty$, but

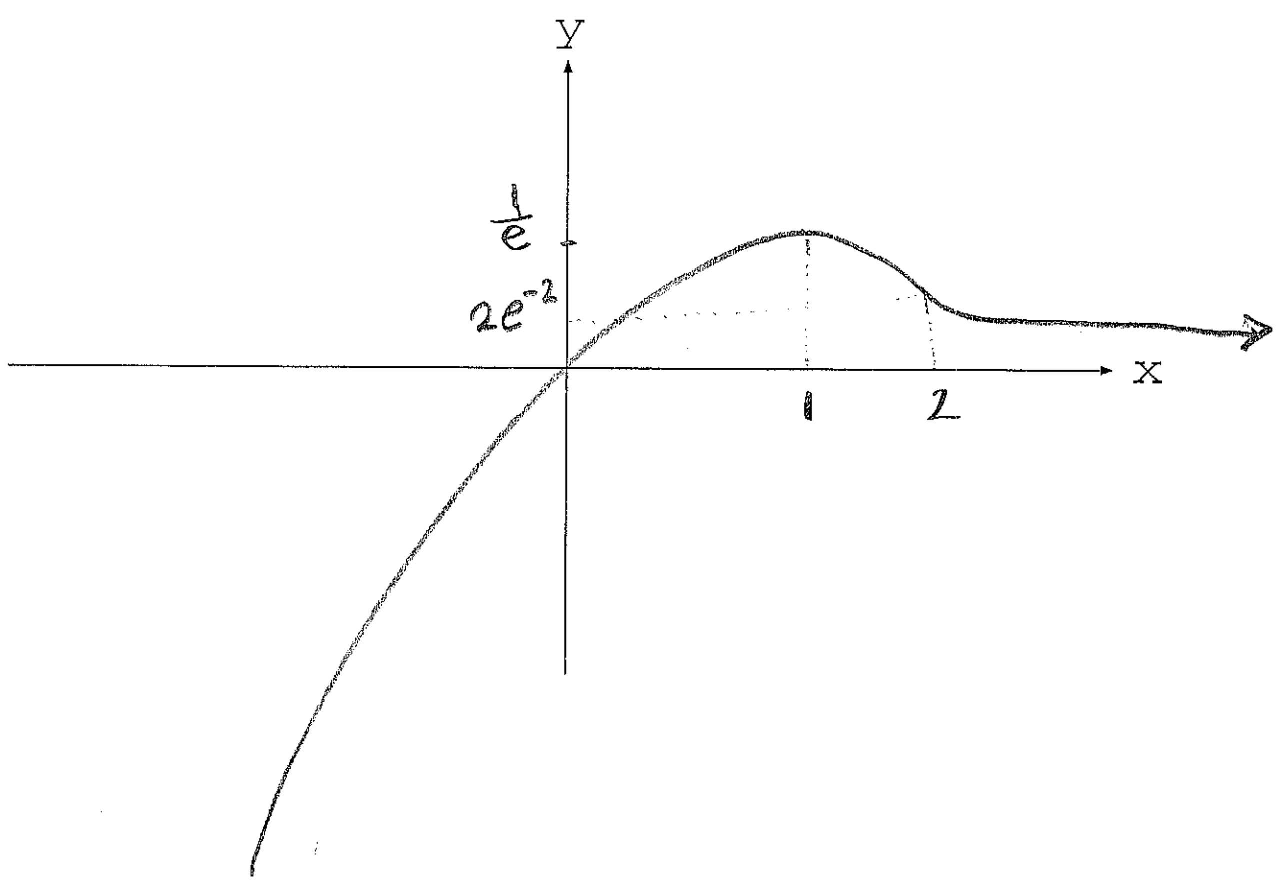
$\lim_{x \to \infty} xe^{-x} = \lim_{x \to \infty} \frac{x}{e^x} = \lim_{x \to \infty} \frac{1}{e^x} = 0.$

$\boxed{y = 0 \text{ is a H.A at } +\infty \text{ only}}$

[1] (d) the x and y intercepts if any

$\boxed{\text{The origin } (0,0) \text{ is the only } x \text{ and } y \text{ intercept.}}$

[5] (ii) Draw a clearly labelled sketch of $y = xe^{-x}$ on the following axes.



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2. *[15 marks]*

If a university charges \$12 for a football ticket, it sells on average 70,000 tickets. For every \$1 increase in the ticket price it loses 2000 in attendance. If every person spends an average of \$3 on refreshments, then what price/ticket should they charge in order to maximize their revenue? How many tickets will they sell at this price, and what will be the maximum revenue?

Let x = the number of \$1 increases

Attendance $= 70{,}000 - 2000x$

revenue per person $= \underbrace{(12+x)}_{\text{ticket}} + \underbrace{3}_{\text{refreshment}} = 15 + x$

Total revenue

$$R = (70{,}000 - 2000x)(15+x)$$

$$\frac{dR}{dx} = -4{,}000x + 40{,}000 = 0 \quad \text{when } x = 10$$

$$\boxed{\text{Ticket Price} = 12 + x = \$22}$$

$$\boxed{\text{No of Tickets} = 70{,}000 - 2000x = 50{,}000}$$

$$\boxed{\text{Total Revenue} = 50{,}000 \times 25 = \$1.25 \text{ million}}$$

Acceptable arguments for R = max at $x = 10$:

1) $R = f(x)$ is an upside down parabola which always has a max at $R' = 0$.

or 2) $\frac{d^2R}{dx^2} = -4000 < 0$ for every value of x.

(It is not enough to note that $\frac{d^2R}{dx^2} < 0$ at $x = 10$. This only says we are at local max.)

or 3) $\frac{dR}{dx} > 0$ for every $x < 10$ and $\frac{dR}{dx} < 0$ for every $x > 10$.

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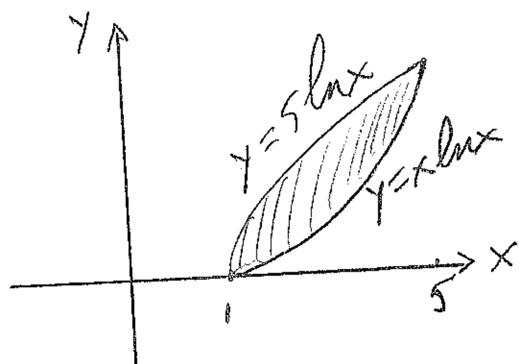
3. *[12 marks]*

Find the area of the region bounded by the curves $y = 5\ln x$ and $y = x\ln x$. [A rough sketch may be helpful, but is **not** required.]

Points of intersection:

$$5\ln x = x\ln x$$
$$(5-x)\ln x = 0 \qquad x=5 \text{ and } x=1$$

Both $f(x) = 5\ln x$ and $g(x) = x\ln x$ are cont. fcns on $[1,5]$ and never equal on $(1,5)$. At $x=e$, for example, $f(x) = 5\ln e = 5$ and $g(x) = e\ln e = e < 5$. Hence $5\ln x$ lies above $x\ln x$ on the entire interval $[1,5]$



This is a <u>rough</u> sketch.

$$\text{Area} = \int_1^5 (5\ln x - x\ln x)\,dx = \boxed{\int_1^5 (5-x)\ln x\,dx}$$

$u = \ln x \qquad dv = 5-x$

$du = \frac{dx}{x} \qquad v = 5x - \frac{x^2}{2}$

$$\boxed{\text{Area} = \left(5x-\frac{x^2}{2}\right)\ln x\Big|_1^5 - \int_1^5 \left(5-\frac{x}{2}\right)dx}$$

$$\text{Area} = \left(25-\frac{25}{2}\right)\ln 5 - \left[5x - \frac{x^2}{4}\right]_1^5$$

$$= \frac{25}{2}\ln 5 - \left[\left(25-\frac{25}{4}\right) - \left(5-\frac{1}{4}\right)\right]$$

$$\boxed{\text{Area} = \frac{25}{2}\ln 5 - 14} \approx 6.12$$

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4. *[15 marks]* Find the following integrals.

[5] (a) $\int_1^e \frac{\ln x}{x}\,dx$

Let $u = \ln x \quad du = \frac{dx}{x}$

$= \int_0^1 u\,du$

$= \frac{u^2}{2}\Big|_0^1 = \boxed{\frac{1}{2}}$

[10] (b) $\int \frac{x^3}{(1+x^2)^2}\,dx$ [Suggestion: $u = 1+x^2$, but there are other ways too; partial fractions is **not** one of these ways.]

1) By Substitution I: Let $u = 1+x^2 \quad \frac{du}{2} = x\,dx$

$\int \frac{x^3}{(1+x^2)^2}\,dx = \int \frac{x^2 \cdot x\,dx}{(1+x^2)^2} = \frac{1}{2}\int \frac{(u-1)\,du}{u^2} = \frac{1}{2}\int \left(\frac{1}{u} - \frac{1}{u^2}\right) du$

$= \frac{1}{2}\left[\ln|u| + \frac{1}{u}\right] + C = \boxed{\frac{1}{2}\left[\ln(1+x^2) + \frac{1}{1+x^2}\right] + C}$

2) By Substitution II: $\int \frac{x^3}{(1+x^2)^2}\,dx = \int \frac{x^3+x-x}{(1+x^2)^2}\,dx = \int \left[\frac{x^3+x}{(1+x^2)^2} - \frac{x}{(1+x^2)^2}\right] dx$

$= \int \left[\frac{x}{1+x^2} - \frac{x}{(1+x^2)^2}\right] dx$ Now $u = 1+x^2$, $\frac{du}{2} = x\,dx$

$= \frac{1}{2}\int \left[\frac{1}{u} - \frac{1}{u^2}\right] du$ Just like before and the answer looks just the same.

3) By parts: Let $u = x^2 \quad dv = \frac{x}{(1+x^2)^2} \quad v = \int \frac{x\,dx}{(1+x^2)^2} = -\frac{1}{2}\,\frac{1}{1+x^2}$

$du = 2x\,dx$

$\int x^2 \cdot \frac{x}{(1+x^2)^2}\,dx = -\frac{1}{2}\,\frac{x^2}{1+x^2} + \int \frac{x\,dx}{1+x^2}$

$= \boxed{-\frac{1}{2}\,\frac{x^2}{1+x^2} + \frac{1}{2}\ln(1+x^2) + C}$

This looks different from the first answer, but the difference $= 1$, which is a constant.