Solved

Department of Mathematics
University of Toronto

WEDNESDAY, February 28, 2007 6:10-8:00 PM

MAT 133Y TERM TEST #3

Calculus and Linear Algebra for Commerce

Duration: 1 hour 50 minutes

Aids Allowed: A non-graphing calculator, with empty memory, to be supplied by student.

Instructions: Fill in the information on this page, and make sure your test booklet contains 11 pages. In addition, you should have a **multiple-choice answer sheet**, on which you should fill in your name, number, tutorial time, tutorial room, and tutor's name.

This test consists of 10 multiple choice questions, and 4 written-answer questions.
For the **multiple choice questions** you can do your rough work in the test booklet, but you must record your answer by circling the appropriate letter **on the answer sheet** with your pencil. Each correct answer is worth 4 marks; a question left blank, or an incorrect answer, or two answers for the same question is worth 0. For the **written-answer questions**, present your solutions in the space provided. The value of each written-answer question is indicated beside it. ENCLOSE YOUR FINAL ANSWER IN A BOX AND WRITE IT IN INK.

TOTAL MARKS: 100

FAMILY NAME: ______________________

GIVEN NAME: ______________________

STUDENT NO: ______________________

SIGNATURE: ______________________

TUTORIAL TIME and ROOM: ______________________

REGCODE and TIMECODE: ______________________

T.A.'S NAME: ______________________

Regcode	Timecode	Room	**Regcode**	Timecode	Room
T0101A	M9A	LM 155	T0501D	W3D	BF 323
T0101B	M9B	SS1074	T0601A	R4A	MP 137
T0101C	M9C	SS1084	T0601B	R4B	RW 143
T0201A	M3A	MS3163	T0701A	F2A	MP 137
T0201B	M3B	WI1017	T0701B	F2B	SS2108
T0201C	M3C	BA2145	T0701C	F2C	LM 162
T0201D	M3D	BA2155	T0801A	F3A	SS1083
T0301A	T3A	SS2110	T0801B	F3B	SS1073
T0301B	T3B	SS1083	T5101A	M5A	SS1069
T0401A	W9A	SS1084	T5101B	M5B	SS2108
T0401B	W9B	SS2127	T5201A	M6A	SS2110
T0501A	W3A	SS2102			
T0501B	W3B	RW 110			
T0501C	W3C	MP 202			

FOR MARKER ONLY	
Multiple Choice	
B1	
B2	
B3	
B4	
TOTAL	

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PART A. Multiple Choice

1. *[4 marks]*

If the sum of two numbers is -4, then the minimum value of the sum of their squares is

A. 10

B. 2

C. 8

D. 5

E. 16

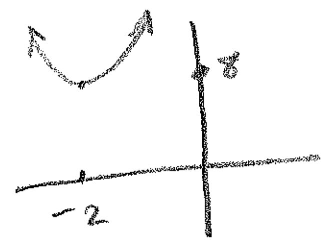
$x+y=-4 \qquad y=-4-x$

$S = x^2+y^2 = x^2+(-4-x)^2 = 2x^2+8x+16$

$\frac{dS}{dx} = 4x+8 = 0 \Rightarrow x=-2 \Rightarrow y=-2$

so $S = (-2)^2+(-2)^2 = \boxed{8}$ C

graph of S



2. *[4 marks]*

The function $f(x) = x^3 - 3x$ on the interval $[-2, 3]$ has

A. one absolute maximum and one absolute minimum

B. one absolute maximum and two absolute minima

C. one absolute maximum and no absolute minimum

D. two absolute maxima and one absolute minimum

E. two absolute maxima and two absolute minima

$f'(x) = 3x^2-3x = 3x(x-1)$

Crit pts are at $x=0$ and $x=1$

f cont. $\Rightarrow$ f has abs. extrema on $[-2,3]$ at some of

$x=0, x=1, x=-2, x=3$

$f(0)=0 \quad \underbrace{f(1)=-2 \quad f(-2)=-2}_{\text{abs min}} \quad \underbrace{f(3)=18}_{\text{abs max}}$

B

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3. *[4 marks]*
$\lim_{x\to 0} \frac{e^x + e^{2x} - 2}{x}$ is

$$\overset{0/0}{=} \lim_{x\to 0} \frac{e^x + 2e^{2x}}{1} = 1+2 = \boxed{3} \text{ E}$$

A. undefined

B. 0

C. 1

D. 2

E. 3

4. *[4 marks]*
$\lim_{x\to\infty} \frac{2^x + x}{2^x + x^2} =$

$$\overset{\infty/\infty}{\lim_{x\to\infty}} \frac{2^x \ln 2 + 1}{2^x \ln 2 + 2x} \overset{\infty/\infty}{=} \lim_{x\to\infty} \frac{2^x (\ln 2)^2}{2^x (\ln 2)^2 + 2} \overset{\infty/\infty}{}$$

$$= \lim_{x\to\infty} \frac{2^x (\ln 2)^3}{2^x (\ln 2)^3} = \boxed{1} \text{ A}$$

A. 1

B. 0

C. $\ln 2$

D. $\frac{(\ln 2)^2}{(\ln 2)^2 + 2}$

E. 2

or (much easier):

$$\lim_{x\to\infty} \frac{x}{2^x} = \lim_{x\to\infty} \frac{1}{2^x \ln 2} = 0$$

So $\lim_{x\to\infty} \frac{2^x + x}{2^x + x^2} = \lim_{x\to\infty} \frac{1 + \frac{x}{2^x}}{1 + \frac{x^2}{2^x}}$

and $\lim_{x\to\infty} \frac{x^2}{2^x} = \lim \frac{2x}{2^x \ln 2} = 0$ as well

This leaves $\frac{1}{1} = \boxed{1}$ A

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5. *[4 marks]*

$$\lim_{x\to 0} \frac{\sqrt{1+x} - 1 - \frac{1}{2}x}{x^2} =$$

A. 0

B. $-\frac{1}{8}$

C. $\frac{1}{8}$

D. $\frac{1}{4}$

E. $-\frac{1}{2}$

$\frac{0}{0}$ $\lim_{x\to 0} \frac{\frac{1}{2(1+x)^{\frac{1}{2}}} - \frac{1}{2}}{2x}$ still $\frac{0}{0}$

$= \lim_{x\to 0} \frac{-\frac{1}{4(1+x)^{3/2}}}{2} = \boxed{-\frac{1}{8}}$ B

6. *[4 marks]*

$$\int \frac{3x^4 + \sqrt{x} + 5}{x}\,dx \text{ is equal to}$$

A. $3(\ln|x|)^4 x + \sqrt{\ln|x|} + 5 + C$

B. $\frac{1}{12}x^4 + \sqrt{x} + \frac{5}{x} + C$

C. $\frac{3}{4}x^4 + 2\sqrt{x} + 5\ln|x| + C$

D. $\frac{1}{12}x^4 + 2\sqrt{x} + 5\ln|x| + C$

E. $3(\ln|x|)^4 + 2\sqrt{x} + \frac{5}{x} + C$

$\int \left(3x^3 + \frac{1}{\sqrt{x}} + \frac{5}{x}\right) dx$

$= \boxed{\frac{3x^4}{4} + 2\sqrt{x} + 5\ln|x| + C}$

C

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7. *[4 marks]*

$$\int_0^1 \frac{x\,dx}{(x^2+1)^4} =$$

A. $\frac{1}{48}$

B. $\frac{7}{48}$

C. $\frac{7}{12}$

D. $\frac{1}{6}$

E. $\frac{7}{24}$

Let $u = x^2+1$

$\frac{1}{2} du = x\,dx$

$$\frac{1}{2}\int_1^2 \frac{du}{u^4} = \frac{1}{2}\left(-\frac{1}{3u^3}\right)\Big|_1^2$$

$$= \frac{1}{6}\left[-\frac{1}{8}+1\right] = \boxed{\frac{7}{48}}$$

B

8. *[4 marks]*

$$\int_0^1 5^x\,dx =$$

A. $\frac{4}{\ln 5}$

B. $\frac{5}{\ln 5}$

C. $\frac{25}{2}$

D. $\frac{15}{2}$

E. 4

$$\frac{5^x}{\ln 5}\Big|_0^1 = \frac{5^1-5^0}{\ln 5} = \boxed{\frac{4}{\ln 5}}$$

A

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9. *[4 marks]*

If $F(x) = \int_1^x \frac{1}{\sqrt{1+t^3}}\,dt$ then $F'(2) =$

A. $\frac{1}{\sqrt{2}}$

B. 1

C. $\frac{1}{\sqrt{1+x^3}} - \frac{1}{\sqrt{2}}$

D. $\frac{1}{3}$

E. $\frac{1}{3} - \frac{1}{\sqrt{2}}$

$F'(x) = \frac{1}{\sqrt{1+x^3}}$

$F'(2) = \frac{1}{\sqrt{1+8}} = \boxed{\frac{1}{3}}$

D

10. *[4 marks]*

After subdividing $[0, 24]$ into $n = 4$ subintervals of equal length, Simpson's estimate for $\int_0^{24} \frac{dx}{1+x}$ is closest to

A. 3.76

B. 3.22

C. 3.95

D. 4.43

E. 4.75

$\Delta x = \frac{24-0}{4} = 6$

0 6 12 18 24

x_0 x_1 x_2 x_3 x_4

$S_4 = \frac{\Delta x}{3}\left[y_0 + 4y_1 + 2y_2 + 4y_3 + y_4\right]$

$S_4 = \frac{6}{3}\left[1 + \frac{4}{7} + \frac{2}{13} + \frac{4}{19} + \frac{1}{25}\right]$

$\approx \boxed{3.95}$ C

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PART B. Written-Answer Questions

1. *[17 marks]*

If $f(x) = \dfrac{4}{x^2(x+3)}$ where

$$f'(x) = \frac{-12(x+2)}{x^3(x+3)^2}$$

and

$$f''(x) = \frac{24(2x^2+8x+9)}{x^4(x+3)^3}$$

[3] (a) find all the horizontal and vertical asymptotes of f

$\boxed{\text{V.A. at } x=0 \text{ and } x=-3}$ and since $\lim\limits_{x\to\infty} f(x) = 0$; $\lim\limits_{x\to-\infty} f(x) = 0$

$\boxed{\text{H.A. at } y=0 \text{ as } x\to\infty \text{ and } x\to-\infty}$

[4] (b) find where f is increasing, decreasing, and all relative maxima or minima

	f'	f
$(-\infty,-3)$	$-$	dec
$(-3,-2)$	$-$	dec
$(-2, 0)$	$+$	inc
$(0,\infty)$	$-$	dec

$\vee$ $\boxed{x=-2 \text{ is a relative min}}$

asymptote at $x=0$

[4] (c) find where f is concave upward, concave downward, and all inflection points

Since $8^2 - 4\cdot2\cdot9 < 0$, $f'' \neq 0$ for any real x.

	f''	f
$(-\infty,-3)$	$-$	Bonc. down
$(-3, 0)$	$+$	Conc. up
$(0,\infty)$	$+$	Conc. up

but no pt. of inflection since $x=-3$ is not a pt. on the graph.

[6] (d) draw a clear graph of f on the axes **on the next page**

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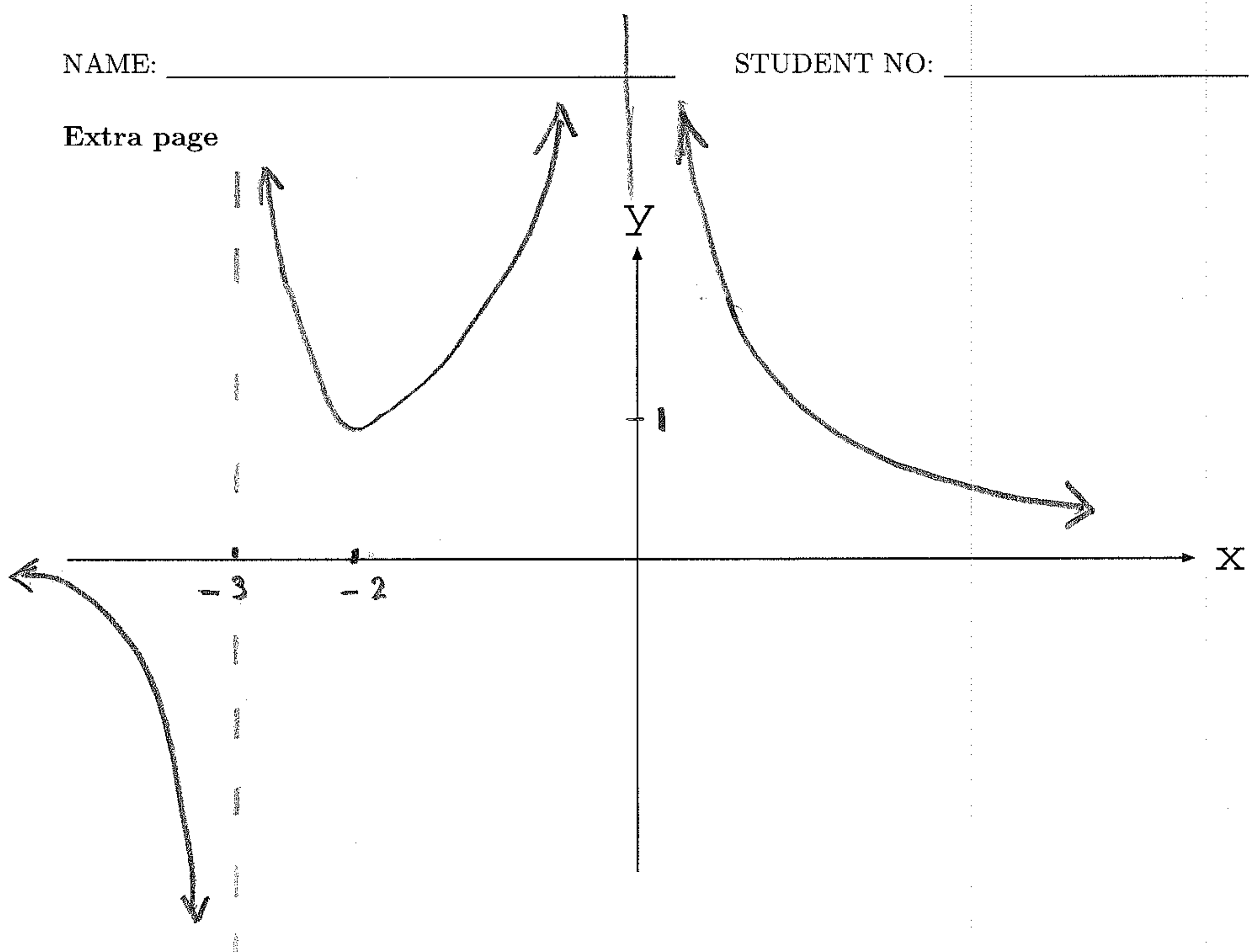
Extra page



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2. *[15 marks]*

If a fast food outlet charges $\$p$ for a hamburger then they will sell $q = \left(\frac{30,000}{p}\right)^{3/2}$ hamburgers/week. If the total cost of producing q hamburgers is $(q + 10,000)$ dollars, then find the price that they should charge/hamburger and the weekly output (q) in order to maximize their profit.

$$\boxed{q^{2/3} = \frac{30,000}{p} \qquad p = \frac{30,000}{q^{2/3}}}$$

$$\text{Profit} = \pi = R - C$$
$$= pq - C$$
$$\pi(q) = 30,000\, q^{\frac{1}{3}} - q - 10,000$$

$$\frac{d\pi}{dq} = 10,000\, q^{-\frac{2}{3}} - 1$$

$\frac{d\pi}{dq} = 0$ when $10,000 = q^{\frac{2}{3}}$

or $\boxed{q = 1,000,000} = 10^6$

and $p = \dfrac{3 \times 10^4}{(10^6)^{\frac{2}{3}}}$ i.e. $\boxed{p = \$3}$

Argument that this is max:

Version 1: $\dfrac{d^2\pi}{dq^2} = -\dfrac{20,000}{3} q^{-5/3} < 0$ for <u>all</u> values of q.

Version 2: $\dfrac{d\pi}{dq} > 0$ $q < 1M$ and $\dfrac{d\pi}{dq} < 0$ $q > 1M$

so π increases until $q = 1M$ and decreases forever after.

Note: Alternative version in terms of p:

$$\pi = \frac{p\,(30,000)^{3/2}}{p^{3/2}} - \frac{(30,000)^{3/2}}{p^{3/2}} - 10,000$$

$$\frac{d\pi}{dp} = (30,000)^{3/2}\left[-\frac{1}{2}p^{-3/2} + \frac{3}{2}p^{-5/2}\right] = \frac{(30,000)^{3/2}}{2p^{5/2}}[3 - p] \begin{cases} > 0, & p < 3 \\ < 0 & p > 3 \end{cases}$$

So $\frac{d\pi}{dp} = 0$ when $\boxed{p = 3}$ and $q = \left(\frac{30,000}{3}\right)^{3/2} = \boxed{1M}$ as before.

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3. *[15 marks]*

A manufacturer's marginal cost function, MC, in dollars, is given by

$$\mathrm{MC} = 2000\frac{\ln q}{q} + 300 \quad = \frac{dC}{dq}$$

when q items are produced and $q \geq 1$. If the cost of producing only 1 item is \$10,000, find the marginal cost and the **average cost** when $q = 100$.

$$C = \int \left[2000 \frac{\ln q}{q} + 300\right] dq$$

$$= \int 2000 \frac{\ln q}{q}\, dq + 300q$$

Let $u = \ln q \qquad du = \frac{dq}{q}$

$$C = \int 2000\, u\, du + 300q = 1000\, u^2 + 300q + K$$

$$C = 1000(\ln q)^2 + 300q + K$$

$$10{,}000 = 300 + K \quad (q = 1)$$

$$K = 9700$$

$$C(q) = 1000(\ln q)^2 + 300q + 9700$$

$$\text{Av Cost (when } q = 100) = \frac{C(100)}{100} = 10(\ln 100)^2 + 300 + 97$$

$$\boxed{\text{Av Cost} = \$609.08}$$

$$\text{MC (when } q = 100) = \frac{2000 \ln 100}{100} + 300$$

$$\boxed{\text{MC } (q = 100) = \$392.10}$$

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4. *[13 marks]*

Make a rough sketch of the area enclosed by the three curves:

1) the y-axis $x = 0$
2) the graph of $y = e^x$
3) the graph of $y = e^{3x-4}$

shading the enclosed area.
Then calculate the area of the shaded region.
Give a **numerical** answer at the end.

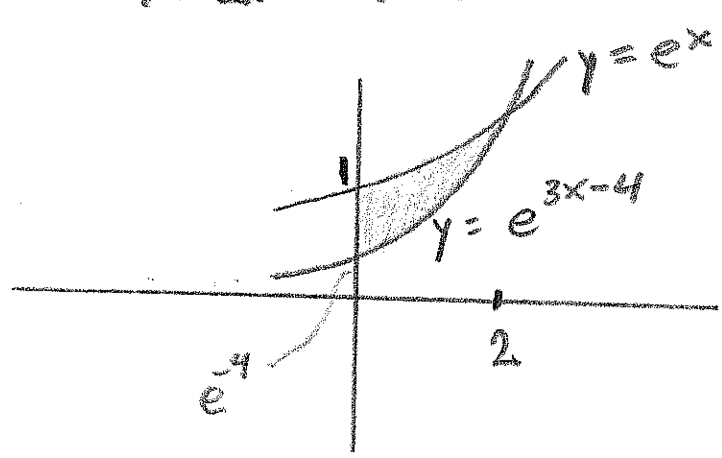
Intersect: $e^x = e^{3x-4}$

$x = 3x - 4$

$4 = 2x \Rightarrow x = 2$

$x = 0 \quad y = e^0 = 1$

$x = 0 \quad y = e^{3 \cdot 0 - 4} = e^{-4}$



$$\text{Area} = \int_0^2 \left[e^x - e^{3x-4}\right] dx$$

$$= \left[e^x - \frac{e^{3x-4}}{3}\right]_0^2$$

$$= \boxed{\left(e^2 - \frac{e^2}{3}\right) - \left(1 - \frac{e^{-4}}{3}\right)}$$

$$= \boxed{\frac{2e^2}{3} - 1 + \frac{e^{-4}}{3}}$$

$$= \boxed{3.932}$$