Solved

Department of Mathematics
University of Toronto

WEDNESDAY, MARCH 8, 2006 6:10-8:00 PM
MAT 133Y TERM TEST #3

Calculus and Linear Algebra for Commerce
Duration: 1 hour and 50 minutes

Aids Allowed: A non-graphing calculator, with empty memory, to be supplied by student.

Instructions: Fill in the information on this page, and make sure your test booklet contains 11 pages. In addition, you should have a **multiple-choice answer sheet**, on which you should fill in your name, number, tutorial time, tutorial room, and tutor's name.

This test consists of 10 multiple choice questions, and 4 written-answer questions.
For the **multiple choice questions** you can do your rough work in the test booklet, but you must record your answer by circling the appropriate letter **on the answer sheet** with your pencil. Each correct answer is worth 4 marks; a question left blank, or an incorrect answer, or two answers for the same question is worth 0. For the **written-answer questions**, present your solutions in the space provided. The value of each written-answer question is indicated beside it. ENCLOSE YOUR FINAL ANSWER IN A BOX AND WRITE IT IN INK.

TOTAL MARKS: 100

FAMILY NAME: ______________________

GIVEN NAME: ______________________

STUDENT NO: ______________________

SIGNATURE: ______________________

TUTORIAL TIME and ROOM: ______________________

REGCODE and TIMECODE: ______________________

T.A.'S NAME: ______________________

Regcode	Timecode	Room	**Regcode**	Timecode	Room
T0101A	M9A	LM 123	T0501D	W3D	UCA101
T0101B	M9B	LM 157	T0601A	R4A	LM 123
T0101C	M9C	WI 523	T0601B	R4B	LM 157
T0201A	M3A	WO 20	T0701A	F2A	LM 157
T0201B	M3B	SS2128	T0701B	F2B	MP 118
T0201C	M3C	WI 524	T0701C	F2C	SS1084
T0201D	M3D	UC 52	T0801A	F3A	MP 118
T0301A	T3A	UC 87	T0801B	F3B	WI 523
T0301B	T3B	UC 256	T5101A	M5A	LM 155
T0401A	W9A	LM 123	T5101B	M5B	WI 523
T0401B	W9B	LM 157	T5201A	M6A	LM 123
T0501A	W3A	UC 244			
T0501B	W3B	UC 328			
T0501C	W3C	UC 52			

FOR MARKER ONLY	
Multiple Choice	
B1	
B2	
B3	
B4	
TOTAL	

NAME: ______________________ STUDENT NO: ______________

PART A. Multiple Choice

1. *[4 marks]*

   If $x + y = 10$, then the largest possible value of xy is

   A. non-existent; there is no largest value.

   B. 16

   C. 21

   D. 25

   E. 30

$xy = P = x(10-x)$

$P' = 10 - 2x = 0$

when $x = 5$

and $\boxed{P = 25}$

$P = \cap$ max when $P' = 0$

2. *[4 marks]*

   If $xy = 10$, where x and y are ≥ 0, then the largest possible value of $x + y$ is

   A. non-existent; there is no largest value.

   B. 11

   C. $2\sqrt{10}$

   D. 25

   E. 7

$x + y = S = x + \frac{10}{x}$

$S' = 1 - \frac{10}{x^2}$

$= \frac{x^2 - 10}{x^2}$

$S' < 0 \quad x < \sqrt{10}$

$S' > 0 \quad x > \sqrt{10}$

$2\sqrt{10}$ is actually the smallest value.

S increases to ∞ as $x \to \infty$

NAME: ______________________ STUDENT NO: ______________

3. *[4 marks]*
The demand function for a product is $p = 242 - 3q$ and the average cost function is $\bar{c} = q + 2 + \dfrac{100}{q}$. What price (to the nearest dollar) should be charged in order to maximize profit?

A. \$227

B. \$152

C. \$40

D. \$122

E. \$30

$$\Pi = pq - C$$
$$= q(242-3q) - [q^2 + 2q + 100]$$
$$\Pi' = 242 - 6q - 2q - 2$$
$$= 240 - 8q = 0 \text{ when}$$
$$q = 30$$
$$\Pi = \wedge \text{ max when } \Pi' = 0.$$
$$p = 242 - 3q = \boxed{\$152}$$

4. *[4 marks]*
$\displaystyle\lim_{x\to\infty} \frac{e^{0.000004x}}{10,000x^4} =$

A. 400

B. e^{400}

C. ∞

D. 0

E. 1

$$\lim_{x\to\infty} \frac{e^{10^{-6}x}}{10^4 x^4} = \lim_{x\to\infty} \frac{10^{-6} e^{10^{-6}x}}{4\times 10^4 \cdot x^3}$$
$$= \dots \lim_{x\to\infty} \frac{10^{-24} e^{10^{-6}x}}{4! \cdot 10^4} \boxed{= \infty}$$

$\frac{\infty}{\infty}$ each time.

NAME: ______________________ STUDENT NO: ______________

5. *[4 marks]*

$\lim_{x \to 0} (3x+5)^{\frac{1}{2x+7}}$

A. does not exist

B. $= 0$

C. $= 1$

D. $= \frac{3}{10}$

E. $= 5^{\frac{1}{7}}$

$(3x+5) \to 5$

$\frac{1}{2x+7} \to \frac{1}{7}$

Answer is $\boxed{5^{\frac{1}{7}}}$

6. *[4 marks]*

$\lim_{x \to \infty} \left(1 + \frac{1}{3x}\right)^{7+2x}$

A. $= 0$

B. $= 1$

C. $= e^7$

D. $= e^{2/3}$

E. $= 7$

1^{∞}

$y = \left(1+\frac{1}{3x}\right)^{7+2x}$

$\ln y = (7+2x) \ln\left(1+\frac{1}{3x}\right)$

$= \dfrac{\ln\left(1+\frac{1}{3x}\right)}{\frac{1}{2x+7}}$ $\quad \frac{0}{0}$

$\lim_{x\to\infty} \ln y = \lim_{x\to\infty} \dfrac{\frac{1}{1+3x} \cdot -\frac{1}{3x^2}}{-\frac{1}{(2x+7)^2} \cdot 2}$

$= \lim_{x\to\infty} \frac{1}{1+3x} \cdot \frac{1}{2} \cdot \frac{(2x+7)^2}{3x^2} = \frac{2}{3}$

so $y \to e^{\frac{2}{3}}$

NAME: ____________________ STUDENT NO: ____________________

7. *[4 marks]*
The area of the region bounded by the x-axis, the vertical lines $x = 1$ and $x = 4$, and the curve $y = -\frac{1}{x}$ is

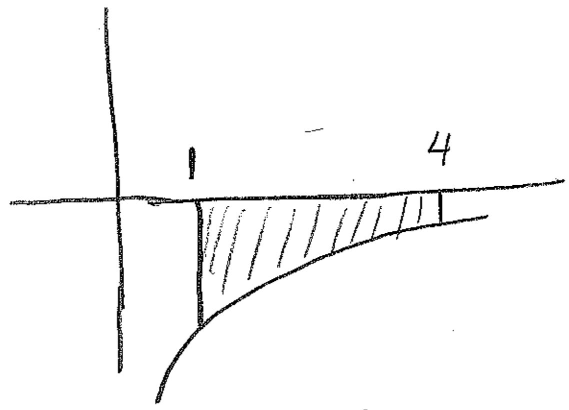
A. 4

B. $-\ln 4$

C. $\ln \frac{1}{4}$

D. $\ln 4$

E. -4



Area $= \int_1^4 [0-(-\frac{1}{x})]dx = \int_1^4 \frac{1}{x}dx$

$= \ln 4 - \ln 1 = \boxed{\ln 4}$

8. *[4 marks]*
$\int_{-2}^{0} \frac{1}{\sqrt{1-4x}}\, dx =$

A. 1

B. 2

C. 3

D. 4

E. 5

$u = 1-4x$
$du = -4dx$

$-\frac{1}{4}\int_9^1 \frac{1}{\sqrt{u}}\, du$

$= \frac{1}{4}\int_1^9 u^{-\frac{1}{2}}du = \frac{2}{4}\sqrt{u}\Big|_1^9 = \frac{1}{2}(3-1) = \boxed{1}$

or directly

$= \frac{2(1-4x)^{\frac{1}{2}}}{-4}\Big|_{-2}^{0}$

$= -\frac{1}{2} + \frac{1}{2}\cdot 9^{\frac{1}{2}} = \boxed{1}$

NAME: ______________________ STUDENT NO: ______________________

9. *[4 marks]*

If the trapezoidal rule, with $n = 3$, is used to estimate the value of

$$\int_1^4 \sqrt{t^3+1}\, dt$$

then the result, T_3, is closest to

x_0	x_1	x_2	x_3	$\Delta x = 1$
1	2	3	4	

A. 26.1

B. 13.0

C. 8.7

D. 4.5

E. 1.5

$$T_3 = \frac{\Delta x}{2}\left[f(x_0) + 2f(x_1) + 2f(x_2) + f(x_3)\right]$$

$$= \frac{1}{2}\left[\sqrt{2} + 2\sqrt{9} + 2\sqrt{28} + \sqrt{65}\right]$$

$$\approx \boxed{13.03}$$

10. *[4 marks]*

If $f(x) = \int_{\sqrt{3}}^{x} \sqrt{u^2+1}\, du$, then $f'(\sqrt{8}) =$

A. 1

B. $\int_{\sqrt{3}}^{\sqrt{8}} \frac{u}{\sqrt{u^2+1}}\, du$

C. 3

D. $\frac{\sqrt{8}}{3} - \frac{\sqrt{3}}{2}$

E. $\frac{\sqrt{8}}{3}$

$$f'(x) = \sqrt{x^2+1}$$

$$f'(\sqrt{8}) = \sqrt{8+1} = \boxed{3}$$

NAME: ____________________ STUDENT NO: ____________________

PART B. Written-Answer Questions

1. *[16 marks]*

Given the function $f(x) = \dfrac{\ln x}{x}$

and it's derivatives $f'(x) = \dfrac{1 - \ln x}{x^2}$

and $f''(x) = \dfrac{2\ln x - 3}{x^3}$

Domain is $(0, \infty)$ only

(a) *[3 marks]*
find the asymptotes, [justifying your answer(s)]

As $x \to 0^+$, $\frac{1}{x} \to \infty$ and $\ln x \to -\infty$

So $\frac{1}{x} \ln x \to -\infty$ and $\boxed{x = 0 \text{ is a V.A.}}$

As $x \to \infty$ $\ln x \to \infty$ and $x \to \infty$, so by L'Hôp.

$\lim\limits_{x \to \infty} \dfrac{\ln x}{x} = \lim\limits_{x \to \infty} \dfrac{\frac{1}{x}}{1} = 0$ and

$\boxed{y = 0 \text{ is a H.A. at } +\infty}$

(b) *[3 marks]*
find the intervals where the function is increasing and decreasing and all relative extrema

Crit pt at $\ln x = 1$, that is, $x = e$

	f'	f
$(0, e)$	$+$	increasing
(e, ∞)	$-$	decreasing

$\wedge$ $\boxed{x = e \text{ is a local max.}}$

There are no mins.

Could also evaluate

$f''(e) = \dfrac{2-3}{e^3} < 0$ so $x = e$ local max

but this is not necessary.

NAME: ____________________ STUDENT NO: ____________________

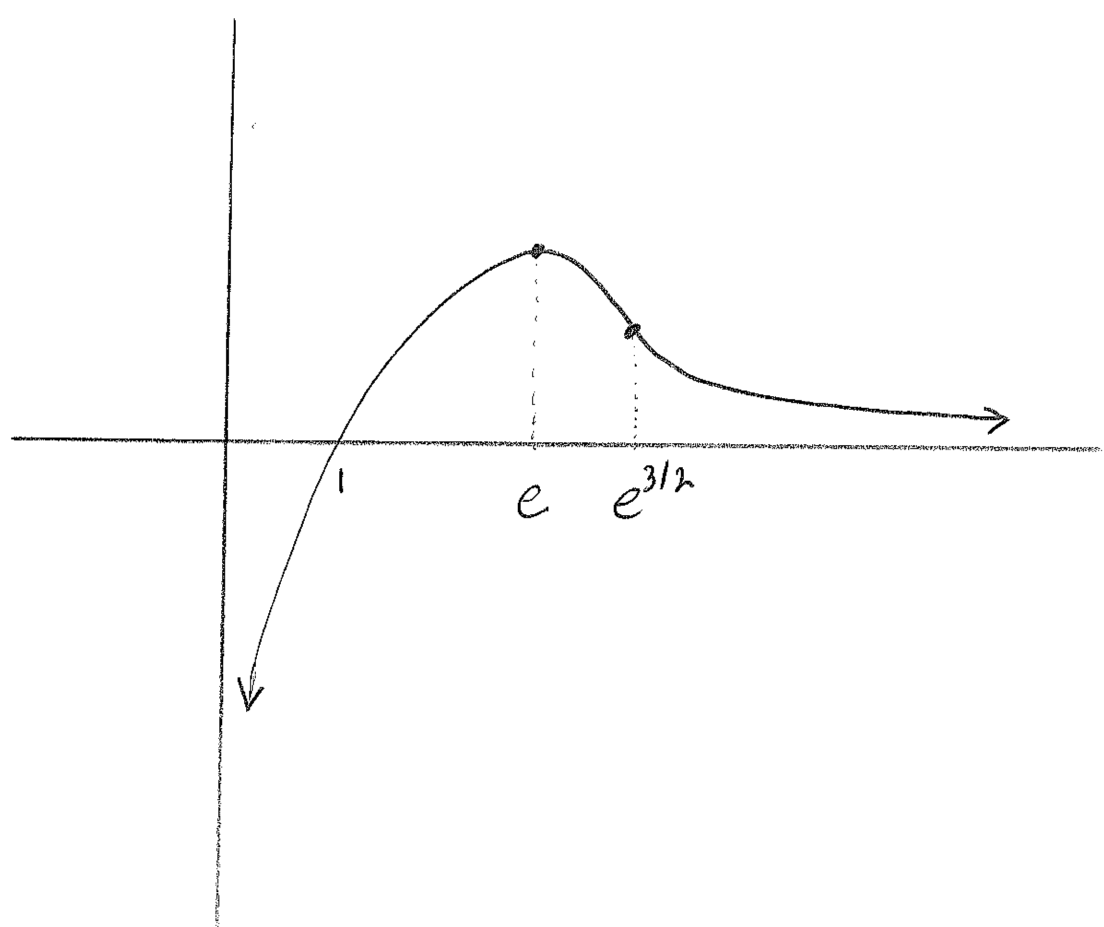
1. (c) *[3 marks]*
find the intervals where the function is concave up and down and all inflection points

$f'' = 0$ when $\ln x = \frac{3}{2}$, i.e. $x = e^{3/2}$

	f''	conc f
$(0, e^{3/2})$	$-$	down
$(e^{3/2}, \infty)$	$+$	up

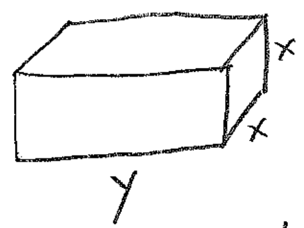
$\boxed{x = e^{3/2} \text{ is a P.O.I.}}$

(d) *[7 marks]*
draw a sketch of the graph of $y = f(x)$.



NAME: ______________________ STUDENT NO: ______________

2. *[14 marks]*
A rectangular wooden crate with two square sides (not the top or the bottom) is to be constructed out of two qualities of wood. The sides and the top are to be made of wood that costs .01 cents/cm ^{2}; the bottom is to be made of wood that costs .02 cents/cm ^{2} If the crate is to hold 274,000 cm ^{3}, what should be its dimensions to minimize the cost?



$x^2 y = 274{,}000 \qquad x > 0$

Let C = cost.

$$C = \underset{\text{bottom}}{.02xy} + 3 \times .01xy + 2 \times .01x^2$$

$$C = .05xy + .02x^2$$

$$C = \frac{.05 \times 274{,}000}{x} + .02x^2$$

$$C' = -\frac{13{,}700}{x^2} + .04x$$

$$= \frac{.04(x^3 - 342{,}500)}{x^2} \qquad x = (342{,}500)^{\frac{1}{3}}$$

$C' = 0$ when $\boxed{x \approx 70 \text{ cm.}}$ and $\boxed{y \approx 56 \text{ cm}}$

C is minimal here because $x^3 < 342{,}500 \Rightarrow C' < 0$
and $x^3 > 342{,}500 \Rightarrow C' > 0$

so C is decreasing from 0 to 70 and increasing <u>forever</u> afterwards.

3. *[15 marks]*
In each part-question, find $y(x)$ which satisfies the given conditions

[7] (a) $\dfrac{dy}{dx} = \dfrac{(\ln x)^2}{x}$ with $y(e) = 1$.

$$y = \int \frac{(\ln x)^2}{x}\,dx \qquad u = \ln x \quad du = \frac{1}{x}\,dx$$

$$= \int u^2\,du = \frac{u^3}{3} + C = \frac{(\ln x)^3}{3} + C$$

When $x = e$ $\quad y = \dfrac{(\ln e)^3}{3} + C = \dfrac{1}{3} + C = 1$

$C = \dfrac{2}{3}$ and $\boxed{y = \dfrac{(\ln x)^3}{3} + \dfrac{2}{3}}$

[8] (b) $\dfrac{d^2y}{dx^2} = e^{2x}$ with $y(0) = 4$, $\dfrac{dy}{dx}(0) = 3$.

$$\frac{dy}{dx} = \frac{1}{2}e^{2x} + C$$

$$3 = \frac{1}{2}e^{0} + C \Rightarrow C = \frac{5}{2}$$

$$\frac{dy}{dx} = \frac{1}{2}e^{2x} + \frac{5}{2}$$

$$y = \frac{1}{4}e^{2x} + \frac{5}{2}x + K$$

$$4 = \frac{1}{4}e^{0} + K \Rightarrow K = \frac{15}{4}$$

$$\boxed{y = \frac{1}{4}e^{2x} + \frac{5}{2}x + \frac{15}{4}}$$

NAME: ______________________ STUDENT NO: ______________

4. *[15 marks]*

Find the following intergrals:

[5] (a) $\int \frac{x^2}{x-3}\,dx$

$$\begin{array}{r} x+3 \\ x-3 \overline{) x^2 } \\ \underline{x^2-3x} \\ 3x \\ \underline{3x-9} \\ 9 \end{array}$$

$= \int \left[x+3+\frac{9}{x-3}\right] dx$

$= \boxed{\frac{x^2}{2}+3x+9\ln|x-3|+C}$

[5] (b) $\int \frac{e^x - e^{-x}}{e^x + e^{-x}}\,dx$ Let $u = e^x + e^{-x}$

$du = (e^x - e^{-x})dx$

$= \int \frac{du}{u} = \ln|u| + C.$

$= \boxed{\ln(e^x+e^{-x})+C}$ (Note: $e^x+e^{-x} > 0$)

[5] (c) $\int \frac{x^5}{(x^2+4)^2}\,dx$. [Hint: Let $u = x^2 + 4$] $du = 2x\,dx$ $x^2 = u-4$

$= \int \frac{(x^2)^2 x\,dx}{(x^2+4)^2} = \frac{1}{2}\int \frac{(u-4)^2}{u^2}\,du$

$= \frac{1}{2}\int \left(1 - \frac{8}{u} + \frac{16}{u^2}\right) du$

$= \frac{1}{2}\left[u - 8\ln u - \frac{16}{u}\right] + C$

$= \boxed{\frac{1}{2}(x^2+4) - 4\ln(x^2+4) - \frac{8}{x^2+4} + C}$

or $\boxed{\frac{1}{2}x^2 - 4\ln(x^2+4) - \frac{8}{x^2+4} + C}$ (absorbing $\frac{1}{2}\cdot 4$ into C.