Solved

Department of Mathematics
University of Toronto

WEDNESDAY, March 2 , 2005 6:10-8:00 PM
MAT 133Y TERM TEST #3

Calculus and Linear Algebra for Commerce
Duration: 2 hours

Aids Allowed: A non-graphing calculator, with empty memory, to be supplied by student.

Instructions: Fill in the information on this page, and make sure your test booklet contains 10 pages. In addition, you should have a **multiple-choice answer sheet**, on which you should fill in your name, number, tutorial time, tutorial room, and tutor's name.

This test consists of 10 multiple choice questions, and 4 written-answer questions.
For the **multiple choice questions** you can do your rough work in the test booklet, but you must record your answer by circling the appropriate letter **on the answer sheet** with your pencil. Each correct answer is worth 4 marks; a question left blank, or an incorrect answer, or two answers for the same question is worth 0. For the **written-answer questions**, present your solutions in the space provided. The value of each written-answer question is indicated beside it. ENCLOSE YOUR FINAL ANSWER IN A BOX AND WRITE IT IN INK.

TOTAL MARKS: 100

FAMILY NAME: ______________________

GIVEN NAME: ______________________

STUDENT NO: ______________________

SIGNATURE: ______________________

TUTORIAL TIME and ROOM: ______________________

REGCODE and TIMECODE: ______________________

T.A.'S NAME: ______________________

Regcode	Timecode	Room	Regcode	Timecode	Room
T0101A	M9A	BF 323	T0501C	W3C	RW 229
T0101B	M9B	LM 123	T0501D	W3D	BA1240
T0101C	M9C	LM 157	T0601A	R4A	LM 157
T0201A	M3A	UCA101	T0701A	F2A	BF 323
T0201B	M3B	SS2106	T0701B	F2B	LM 157
T0201C	M3C	WB 119	T0701C	F2C	MP 118
T0201D	M3D	LM 157	T0801A	F3A	WA 142
T0301A	T3A	MP 137	T0801B	F3B	WI 523
T0301B	T3B	UC 328	T5101A	R5A	LM 155
T0301C	T3C	MP 134	T5101B	R5B	LM 157
T0401A	W9A	BF 323	T5201A	R6A	SS2111
T0401B	W9B	LM 123			
T0501A	W3A	LM 123			
T0501B	W3B	UC 328			

FOR MARKER ONLY	
Multiple Choice	
B1	
B2	
B3	
B4	
TOTAL	

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PART A. Multiple Choice

1. *[4 marks]*

   If $F'(x) = (x^2+1)\cdot x$ and $F(0) = 1$, then $F(x) =$

   A. $\frac{1}{4}x^4 + \frac{1}{2}x^2$

   B. $\frac{1}{3}x^3 + x + C$

   C. $x^4 + x^2 + 1$

   D. $\frac{1}{4}x^4 + \frac{1}{2}x^2 + 1$

   E. $(\frac{1}{3}x^3 + x)\cdot\frac{1}{2}x^2 + 1$

Either directly, $F(x) = \int (x^2+1)x\,dx$

$= \frac{1}{2}\frac{(x^2+1)^2}{2} + C$

$1 = F(0) = \frac{1}{4} + C \Rightarrow C = \frac{3}{4}$

$F(x) = \frac{1}{4}(x^2+1)^2 + \frac{3}{4}$

$= \boxed{\frac{x^4}{4} + \frac{x^2}{2} + 1}$

or $F'(x) = x^3 + x$

$F(x) = \frac{x^4}{4} + \frac{x^2}{2} + C$

$1 = F(0) = C$

$\boxed{F(x) = \frac{x^4}{4} + \frac{x^2}{2} + 1}$

2. *[4 marks]*

   If $f''(x) = \frac{2}{x^3}$, $f'(1) = -1$, and $f(1) = 2$, then $f(\frac{1}{2}) =$

   A. -1

   B. 2

   C. 0

   D. 3

   E. 1

$f'(x) = -\frac{1}{x^2} + C$

$-1 = f'(1) = -1 + C \Rightarrow C = 0$

$f'(x) = -\frac{1}{x^2}$

$f(x) = \frac{1}{x} + K$

$2 = f(1) = 1 + K \Rightarrow K = 1$

$f(x) = \frac{1}{x} + 1$

$f(\frac{1}{2}) = 2 + 1 = \boxed{3}$

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3. *[4 marks]*

If $F(x) = \int_{-1}^{x} \frac{1}{t^2+t+1}\, dt$, then $F'(2)$

$F'(x) = \frac{1}{x^2+x+1}$

$F'(2) = \frac{1}{4+2+1} = \boxed{\frac{1}{7}}$

A. $= \int_{-1}^{2} \frac{1}{x^2+x+1}\, dx$

B. $= \frac{1}{7}$

C. does not exist

D. cannot be determined

E. $= -\frac{3}{64}$

4. *[4 marks]*

Let $F(x) = \frac{3\ln x \cdot e^x + x^2}{(x^2+1)^3}$. Then $\int_{1}^{2} F'(x)\, dx = F(2) - F(1)$

$= \frac{3(\ln 2)e^2 + 4}{5^3} - \frac{1}{8}$

$\approx \boxed{.0299}$

A. 0.0299

B. 0.0498

C. 0

D. -0.49

E. 102.37

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5. *[4 marks]*
$\lim_{x\to 0} \frac{x+1-e^x}{x^2}$
A. $= 0$
B. $= 1$
C. $= -\frac{1}{2}$
D. $= \frac{1}{2}$
E. does not exist

$\frac{0}{0}$ so $= \lim_{x\to 0} \frac{1-e^x}{2x}$

still $\frac{0}{0}$ $= \lim_{x\to 0} -\frac{e^x}{2} = \boxed{-\frac{1}{2}}$

6. *[4 marks]*
$\lim_{x\to\infty} (1+e^x)^{-e^{-x}} =$
A. $+\infty$
B. e
C. 0
D. -1
E. 1

Let $y = (1+e^x)^{-e^{-x}}$

$\ln y = -e^{-x} \ln(1+e^x)$

$= -\frac{\ln(1+e^x)}{e^x} = -\frac{\infty}{\infty}$ as $x \to \infty$

so $\lim_{x\to\infty} \ln y = \lim_{x\to\infty} -\frac{\frac{1\cdot e^x}{(1+e^x)}}{e^x}$

$= \lim_{x\to\infty} -\frac{1}{1+e^x} = 0$

since $\ln y \to 0$,
$y \to \boxed{1}$

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7. *[4 marks]*
$$\int_1^e \frac{2^{\ln x}}{x}\,dx =$$

A. $\frac{2}{e} - 1$

B. 0

C. $\frac{1}{\ln 2}$

D. 1

E. $\ln 2$

$\int_0^1 2^u\,du$

Let $u = \ln x$.

$du = \frac{1}{x}\,dx$

$x = 1 \Rightarrow u = 0$

$x = e \Rightarrow u = 1$

$= \left.\frac{2^u}{\ln 2}\right|_0^1$

$= \frac{2}{\ln 2} - \frac{1}{\ln 2}$

$= \boxed{\frac{1}{\ln 2}}$

8. *[4 marks]*
$$\int_{-2}^{1} |x+1|\,x\,dx =$$

A. $\frac{3}{2}$

B. $-\frac{1}{6}$

C. $-\frac{2}{3}$

D. $\frac{5}{6}$

E. $\frac{1}{2}$

$\int_{-2}^{-1} -(x+1)x\,dx + \int_{-1}^{1} x(x+1)\,dx$

$= \left[-\frac{x^3}{3} - \frac{x^2}{2}\right]_{-2}^{-1} + \left[\frac{x^3}{3} + \frac{x^2}{2}\right]_{-1}^{1}$

$= \left(\frac{1}{3} - \frac{1}{2}\right) - \left(\frac{8}{3} - 2\right) + \left(\frac{1}{3} + \cancel{\frac{1}{2}}\right) - \left(-\frac{1}{3} + \cancel{\frac{1}{2}}\right)$

$= -\frac{5}{3} - \frac{1}{2} + 2 = \frac{1}{3} - \frac{1}{2} = \boxed{-\frac{1}{6}}$

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9. *[4 marks]*

If a and b are real numbers such that $a < b$ and the function f is differentiable on the whole real line, then the average value of f' on the interval $[a, b]$ equals

A. $\dfrac{f'(a) + f'(b)}{2}$

B. the slope of the line which is tangent, at $\left(\frac{a+b}{2}, f\left(\frac{a+b}{2}\right)\right)$, to the graph of f

C. $\dfrac{f(b) - f(a)}{b - a}$

D. $\dfrac{f(a) + f(b)}{2}$

E. $\dfrac{f'(b) - f'(a)}{b - a}$

$$\overline{f'} = \frac{1}{b-a}\int_a^b f'(x)\,dx$$

$$= \boxed{\frac{f(b) - f(a)}{b-a}}$$

10. *[4 marks]*

$\displaystyle\int_{-\infty}^{0} x(5x^2+1)^{-\frac{3}{2}}\,dx$

A. $= -1$

B. $= -\dfrac{1}{5}$

C. $= 1$

D. diverges

E. $= -\dfrac{1}{3}$

$$= \lim_{R\to\infty} \int_{-R}^{0} x(5x^2+1)^{-3/2}\,dx$$

$$= \lim_{R\to\infty} \int_{5R^2+1}^{1} u^{-\frac{3}{2}}\,\frac{du}{10}$$

(with $u = 5x^2+1$ $du = 10x\,dx$)

$$= \lim_{R\to\infty} \frac{1}{10}\, u^{-\frac{1}{2}}\cdot(-2)\Big|_{5R^2+1}^{1}$$

$$= \lim_{R\to\infty} -\frac{1}{5}\left[\frac{1}{\sqrt{1}} - \frac{1}{\sqrt{5R^2+1}}\right]$$

$$= \boxed{-\frac{1}{5}}$$

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PART B. Written-Answer Questions

1. *[15 marks]*
The owner of an apple orchard estimates that if he plants 24 trees per acre, then the number of apples he gets from each tree is 600 per year. For each additional tree he plants per acre, the number of apples he gets from each tree decreases by 12/year. How many trees should he plant per acre in order to yield the most apples?

Let x = no. of trees per acre above 24.
Le y = total yield/acre per yr.

yield per tree $= 600 - 12x$ $(\Rightarrow 0 \leq x \leq 50)$
trees per acre $= 24 + x$

$y = (24+x)(600-12x)$
$= 12(24+x)(50-x)$

$y' = 12[(50-x) - (24+x)]$ (product rule)

$y' = 12(26-2x) = 24(13-x)$

$y' = 0$ when $\boxed{x = 13 \text{ and the no. of trees/acre} = 37}$

This is a max for y because

a) $y(13) = 12 \cdot 37 \cdot 37 = 16{,}428$
while $y(50) = 0$
and $y(0) = 12 \cdot 24 \cdot 50 = 14{,}400$,
$y(13)$ is the biggest of these 3 and
y is cont. on $[0, 50]$.

or
b) y is a quadratic with negative x^2 term
so max is at $y' = 0$.

or
c) $y' > 0$, $x < 13$ and $y' > 0$, $x > 13$
so $y \uparrow$ $x < 13$ and $y \downarrow$ $x > 13$.

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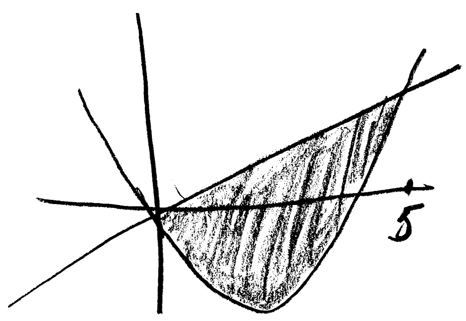
2. Find the total area of the finite region(s) bounded by the curves

[9] (a) $y = x$ and $y = x^2 - 4x$.



$x = x^2 - 4x$ for intersect.

$0 = x^2 - 5x$

$0 = x(x-5)$

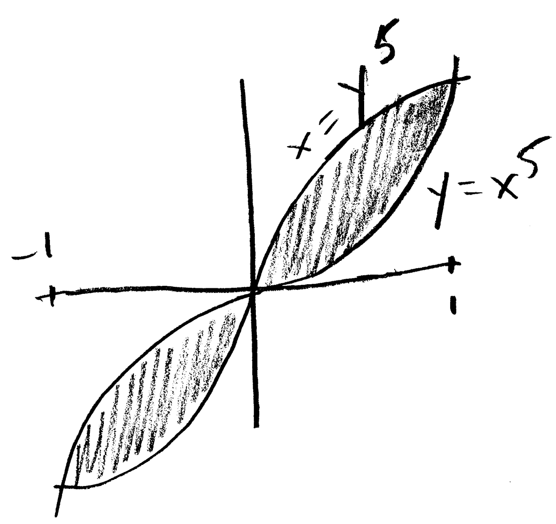
$$A = \int_0^5 [x - (x^2 - 4x)]\,dx$$

$$= \int_0^5 (5x - x^2)\,dx = \left[\frac{5x^2}{2} - \frac{x^3}{3}\right]_0^5$$

$$= \frac{125}{2} - \frac{125}{3} = \boxed{\frac{125}{6}}$$

≈ 20.83

[9] (b) $y = x^5$ and $x = y^5$.



By symmetry, intersect at -1, 0, and 1 and the area is the same from -1 to 0 as it is from 0 to 1.

Also, if $x = y^5$,

$y = x^{\frac{1}{5}}$

So $$A = 2\int_0^1 (x^{\frac{1}{5}} - x^5)\,dx$$

$$= 2\left[\frac{5x^{\frac{6}{5}}}{6} - \frac{x^6}{6}\right]_0^1$$

$$= 2\left[\frac{5}{6} - \frac{1}{6}\right] = \boxed{\frac{4}{3}}$$

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3. *[15 marks]*

At time $t = 0$ an account is opened and at time $t \geq 0$ (t in years), cash flows into the account at the rate $1000t$ dollars per year. If annual interest is 5% compounded continuously, how much money is in the account at the end of 10 years?

$$\text{Future Value} = \int_0^{10} 1000t\, e^{.05(10-t)}\, dt$$

$$= 1000e^{\frac{1}{2}} \int_0^{10} t e^{-.05t}\, dt$$

Let $u = t$ $\quad dv = e^{-.05t}\, dt$

$du = dt$ $\quad v = -\dfrac{e^{-.05t}}{.05}$

$$\text{F.V.} = 1000\, e^{\frac{1}{2}} \left[-20t e^{-.05t} \Big|_0^{10} + 20 \int_0^{10} e^{-.05t}\, dt \right]$$

$$= 1000e^{\frac{1}{2}} \left[-200e^{-\frac{1}{2}} - 400\, e^{-.05t} \Big|_0^{10} \right]$$

$$= 200{,}000e^{\frac{1}{2}} \left[-e^{-\frac{1}{2}} - 2(e^{-\frac{1}{2}} - 1) \right]$$

$$= 200{,}000\, e^{\frac{1}{2}} \left[2 - 3e^{-\frac{1}{2}} \right]$$

$$= \boxed{200{,}000\, (2\sqrt{e} - 3)}$$

$$\approx \boxed{\$\, 59{,}488.51}$$

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4. *[12 marks]*

Find the integral, expressing your final answer as a decimal, accurate to 4 places.

$$\int_0^2 \frac{q^3}{(q+1)^2\ (q+2)}\,dq$$

[You may use the following fact if you find it helpful.]

$$(q+1)^2\ (q+2) = q^3 + 4q^2 + 5q + 2$$

$$q^3+4q^2+5q+2 \,\overline{)\, q^3 \quad}\;1$$
$$\underline{q^3+4q^2+5q+2}$$
$$-4q^2-5q-2$$

$$\int_0^2 \frac{q^3}{(q+1)^2(q+2)}\,dq = \int_0^2 \left[1 - \frac{4q^2+5q+2}{(q+1)^2(q+2)}\right] dq$$

$$\frac{A}{(q+1)^2} + \frac{B}{q+1} + \frac{C}{q+2} \qquad A(q+2) + B(q+1)(q+2) + C(q+1)^2 = 4q^2+5q+2$$

$q=-1 \Rightarrow A=1$; $q=-2 \Rightarrow C=8$

$q=0 \Rightarrow 2A+2B+C=2 \Rightarrow B=-4$

Integral $= \int_0^2 \left[1 - \frac{1}{(q+1)^2} + \frac{4}{q+1} - \frac{8}{q+2}\right] dq$

$= \left[q + \frac{1}{q+1} + 4\ln|q+1| - 8\ln|q+2|\right]_0^2$

$= (2-0) + (\frac{1}{3}-1) + (4\ln 3 - 0) - (8\ln 4 - 8\ln 2)$

$= \boxed{\frac{4}{3} + 4\ln 3 - 8\ln 2} = \boxed{\frac{4}{3} + 4\ln\left(\frac{3}{4}\right)}$

$\approx .182605, \ldots$

$= \boxed{.1826}$ to 4 decimal places.