Solved

Department of Mathematics
University of Toronto

WEDNESDAY, MARCH 3, 2004 6:10-8:00 PM
MAT 133Y TERM TEST #3

Calculus and Linear Algebra for Commerce
Duration: 1 hour 50 minutes

Aids Allowed: A non-graphing calculator, with empty memory, to be supplied by student.

Instructions: Fill in the information on this page, and make sure your test booklet contains 12 pages. In addition, you should have a **multiple-choice answer sheet**, on which you should fill in your name, number, tutorial time, tutorial room, and tutor's name.

This test consists of 10 multiple choice questions, and 4 written-answer questions.
For the **multiple choice questions** you can do your rough work in the test booklet, but you must record your answer by circling the appropriate letter **on the answer sheet** with your pencil. Each correct answer is worth 4 marks; a question left blank, or an incorrect answer, or two answers for the same question is worth 0. For the **written-answer questions**, present your solutions in the space provided. The value of each written-answer question is indicated beside it. ENCLOSE YOUR FINAL ANSWER IN A BOX AND WRITE IT IN INK.

TOTAL MARKS: 100

FAMILY NAME: ________________

GIVEN NAME: ________________

STUDENT NO: ________________

SIGNATURE: ________________

TUTORIAL TIME and ROOM: ________________

REGCODE and TIMECODE: ________________

T.A.'S NAME: ________________

Regcode	Timecode	Room	Regcode	Timecode	Room
T0101A	M9A	SS2130	T0501C	W3C	SS2111
T0101B	M9B	BF 215	T0601A	R4A	SS1088
T0101C	M9C	LM 157	T0601B	R4B	RW 142
T0201A	M3A	SS2111	T0601C	R4C	LM 157
T0201B	M3B	LM 123	T0701A	F2A	BF 323
T0201C	M3C	RW 143	T0701B	F2B	LM 157
T0201D	M3D	MP 134	T0701C	F2C	SS2111
T0301A	T3A	RW 142	T0801A	F3A	RW 142
T0301B	T3B	SS2111	T0801B	F3B	LM 157
T0301C	T3C	MP 134	T5101A	R5A	SS1088
T0401A	W9A	BF 215	T5201A	R6A	MB128
T0401B	W9B	LM 157			
T0501A	W3A	LM 157			
T0501B	W3B	LM 123			

FOR MARKER ONLY	
Multiple Choice	
B1	
B2	
B3	
B4	
TOTAL	

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PART A. Multiple Choice

1. *[4 marks]*
Let M be the absolute maximum of

$$h(x) = 2x^3 - 3x^2 - 12x + 27$$

on $[-3,\ 3]$ and let m be the absolute minimum of the same function on the same interval. Then $m + M =$

$h'(x) = 6x^2 - 6x - 12 = 6(x^2 - x - 2) = 6(x-2)(x+1)$

A. 41

crit pts at $x = -1$ and $x = 2$.

B. 25

$h(-3) = -18 = m$

C. -11

$h(-1) = 34 = M$ $\quad m + M = 16$

D. 0

$h(2) = 7$ $\quad$ E

E. 16

$h(3) = 18$

2. *[4 marks]*
Assume k is some constant. The function

$$g(x) = x^4 - 6kx^2 + 6k^2x$$

has at least one point of inflection

$g'(x) = 4x^3 - 12kx + 6k^2$

$g''(x) = 12x^2 - 12k = 12(x^2 - k)$

A. if and only if $k > 0$

B. if and only if $k \geq 0$

C. for any k

D. if and only if $k < 0$

E. for no value of k

If $k \leq 0$ $g''(x) \geq 0$, hence never changes sign, so no points of inflection.

If $k > 0$, $g''(x) = 12(x - \sqrt{k})(x + \sqrt{k})$, changes signs (twice actually) at least once, hence has at least one point of inflection.

A

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3. *[4 marks]*
If the weekly sales volume (q) for a product is given by $q = \dfrac{32{,}000}{(p+8)^{2/5}}$, where p is the price in dollars, then the approximation **found by using differentials** for the change in sales if the price is raised from \$24.00 to \$24.50 is

A. -50

B. -49

C. -800

D. -51

E. 50

$$dq = \frac{-\frac{2}{5} \times 32{,}000}{(p+8)^{7/5}}\, dp \qquad dp = \frac{1}{2}$$

$$dq = \frac{-\frac{2}{5} \times 32{,}000 \times \frac{1}{2}}{(32)^{7/5}}$$

$$= \frac{-\frac{1}{5} \times 32{,}000}{2^7}$$

$$= \frac{-\frac{1}{5} \times 1000}{2^2} = -50$$

A

4. *[4 marks]*
If $x_1 = 2$ is used as a first estimate for a root of the equation $e^x = x^3$, then to 2 decimal places, Newton's method gives the second estimate $x_2 =$

A. 1.87

B. 1.85

C. 1.86

D. 1.88

E. 1.89

$f(x) = e^x - x^3$ or $x^3 - e^x$

In either case,

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

$$x_{n+1} = x_n - \frac{e^{x_n} - x_n^3}{e^{x_n} - 3x_n^2}. \quad \text{If } x_1 = 2,$$

$$x_2 = 2 - \frac{e^2 - 8}{e^2 - 12} \approx 1.8675$$

A

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5. *[4 marks]*

$$\lim_{x\to 0^+} \frac{1}{x} \ln\left(\frac{1+x}{1-x}\right)$$

A. $= 0$

B. $= 1$

C. $= e^2$

D. $= 2$

E. is undefined

$$= \lim_{x\to 0^+} \frac{\ln(1+x) - \ln(1-x)}{x} \qquad \frac{0}{0} \text{ type}$$

$$= \lim_{x\to 0^+} \frac{\frac{1}{1+x} + \frac{1}{1-x}}{1} = 2 \quad D$$

More complicated version:

$$\lim_{x\to 0^+} \frac{\ln\left(\frac{1+x}{1-x}\right)}{x} \qquad \frac{0}{0} \text{ type}$$

$$= \lim_{x\to 0^+} \frac{\frac{1-x}{1+x} \cdot \left[\frac{(1-x)-(1+x)(-1)}{(1-x)^2}\right]}{1} = \lim_{x\to 0^+} \frac{2}{(1+x)(1-x)}$$

$$= 2 \quad D$$

6. *[4 marks]*

$$\lim_{x\to 1} x^{\left(\frac{1}{\sqrt{x}-1}\right)} =$$

A. 2

B. e^2

C. $e^{-\frac{1}{2}}$

D. $e^{\frac{1}{2}}$

E. $\frac{1}{2}$

1^∞ type

Let $y = x^{\frac{1}{\sqrt{x}-1}}$ $\qquad \ln y = \frac{\ln x}{\sqrt{x}-1}$

$$\lim_{x\to 1} \ln y = \lim_{x\to 1} \frac{\ln x}{\sqrt{x}-1} \qquad \frac{0}{0} \text{ type}$$

$$= \lim_{x\to 1} \frac{\frac{1}{x}}{\frac{1}{2\sqrt{x}}} = 2$$

So $\lim_{x\to 1} y = e^2$ B

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7. *[4 marks]*

If $F(x) = \int_3^x \ln(t^2+1)\,dt$, then $F'(4)$ is closest to

A. 2.83

B. 0.53

C. 0.00

D. 0.47

E. -0.13

$F'(x) = \ln(x^2+1)$

$F'(4) = \ln 17 \approx 2.83$ A

8. *[4 marks]*

$$\int_0^1 \frac{x-1}{x+1}\,dx$$

A. $1 + 2\ln 2$

B. $-\frac{1}{3}$

C. $1 - \ln 4$

D. $-\frac{2}{7}$

E. $-\frac{1}{2}\ln 2$

$$\frac{x-1}{x+1} = 1 - \frac{2}{x+1}$$

$$\int_0^1 \frac{x-1}{x+1}\,dx = \int_0^1 \left[1 - \frac{2}{x+1}\right]dx$$

$$= \Big[x - 2\ln|x+1|\Big]_0^1 = 1 - 2\ln 2$$

$$= 1 - \ln 2^2 = 1 - \ln 4 \quad C$$

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9. *[4 marks]*

$$\int_1^8 \frac{3dx}{x^{\frac{2}{3}}\left(x^{\frac{1}{3}}+1\right)^2} =$$

[Hint: Try a substitution.]

A. $\frac{3}{2}$

B. 96

C. $\frac{63}{8}$

D. $-\frac{3}{2}$

E. $\frac{1}{6}$

Let $u = x^{\frac{1}{3}}+1 \quad du = \frac{1}{3x^{\frac{2}{3}}}dx$

$3du = \frac{dx}{x^{2/3}}$

$u=2$ when $x=1$
$u=3$ when $x=8$

$$\int_1^3 \frac{3dx}{x^{2/3}(x^{\frac{1}{3}}+1)^2} = \int_2^3 \frac{9du}{u^2}$$

$$= -\frac{9}{u}\Big|_2^3 = 9\left(\frac{1}{2}-\frac{1}{3}\right) = \frac{9}{6} = \frac{3}{2} \quad \text{A}$$

10. *[4 marks]*
The area bounded by the graphs of $y = x^2 - 1$ and $y = 2x + 2$ is

A. 4

B. $\frac{32}{3}$

C. $\frac{34}{3}$

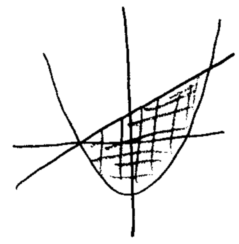
D. $\frac{45}{2}$

E. $\frac{57}{2}$

$x^2 - 1 = 2x + 2$

$x^2 - 2x - 3 = 0 \quad (x-3)(x+1) = 0$

Intersect at $x = -1$ and $x = 3$.



$$\text{Area} = \int_{-1}^{3} [(2x+2)-(x^2-1)]\,dx$$

$$= \int_{-1}^{3} (2x - x^2 + 3)\,dx = \left(x^2 - \frac{x^3}{3} + 3x\right)\Big|_{-1}^{3}$$

$$= (9-9+9) - \left(1+\frac{1}{3}-3\right) = 9 - \left(-\frac{5}{3}\right)$$

$$= \frac{32}{3} \quad \text{B}$$

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PART B. Written-Answer Questions

1. *[15 marks]*

Let $f(x) = \ln|x^3+1|$.

You may assume that $f'(x) = \dfrac{3x^2}{x^3+1}$ and $f''(x) = \dfrac{3x(2-x^3)}{(x^3+1)^2}$.

[10] (a) Find the critical points, intervals where the function is increasing or decreasing, local extrema, intervals where the function is concave up or down, inflection points, and asymptotes, if any.

$\lim_{x\to\pm\infty} f(x) = \infty$ so $\boxed{\text{no H.A.}}$ (Since $|x_3+1| \to \infty$, $\ln|x^3+1| \to \infty$).

$\lim_{x\to -1} f(x) = -\infty$, $\boxed{x=-1 \text{ is a V.A.}}$

$f'(x) = 0$ at $x=0$ and fails to exist at $x=-1$ (where f fails to exist as well).

	f'	f
$(-\infty,-1)$	$-$	dec
$(-1, 0)$	$+$	inc
$(0,\infty)$	$+$	inc.

$\boxed{f \text{ has no local extrema}}$

decreasing on $(-\infty,-1)$

increasing on $(-1,0)$ and $(0,\infty)$

(better " : increasing on $(-1,\infty)$.

$f''(x)=0$ at $x=0$ and $x=2^{\frac{1}{3}}$ and fails to exist at $x=-1$ (as does f).

	f''	f
$(-\infty,-1)$	$-$	conc down
$(-1, 0)$	$-$	conc down
$(0, 2^{\frac{1}{3}})$	$+$	conc. up
$(2^{\frac{1}{3}},\infty)$	$-$	conc. down

$\left.\begin{array}{l} x=0 \\ x=2^{\frac{1}{3}} \end{array}\right\}$ points of inflection

Question continues on Page 8

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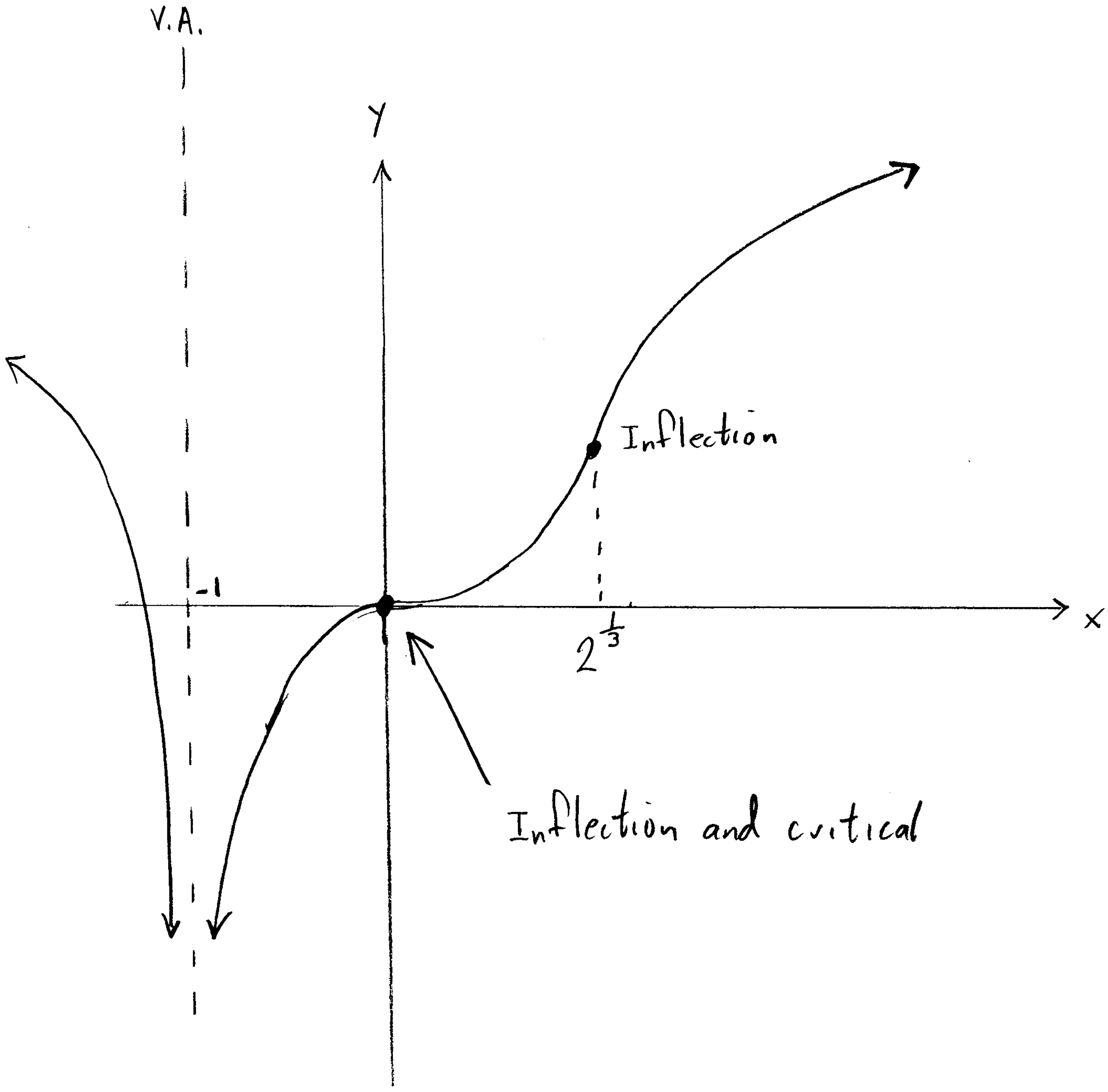
[5] 1. (b) Draw a careful sketch of the graph of $y = f(x)$, labelling the important features.



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2. *[10 marks]*

A certain tavern sells beer only, no wine or spirits, and has seating for 100 customers. If it charges \$3 per beer it will be full but each 10¢ increase in the price of a beer results in one empty seat. Find the price per beer that the tavern should charge to maximize its revenue. You may assume that the time required to drink a beer is the same for all customers and is independent of the price per beer. Show that the price you have found actually **maximizes** revenue.

Let x = number of 10¢ increases, so

$$p = 3 + .10x$$
$$q = 100 - x$$
$$R = pq = (3 + .10x)(100 - x)$$

$0 \leq x \leq 100$ a closed bounded interval.

$$\frac{dR}{dx} = .10(100-x) - (3 + .10x)$$
$$= 7 - .20x$$

$\frac{dR}{dx} = 0$ when $x = 35$ so $\boxed{p = \$6.50}$

Arguments for max: Options

1) $R'' = -.2 < 0$ for every x

or 2) R is a quadratic with negative coefficient of x^2 so critical point is absolut max.

or 3) Since we have a cont. fcn on a closed, bounded interval

$x = 0$ $R(0) = 300$

$x = 35$ $R(35) = 65 \times 6.50 = 422.50$ max

$x = 100$ $R(100) = 0$

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3. *[15 marks]*

The demand function for a product is given by $p = 100e^{-0.1q}$

[6] *(a)* For what values of q is demand elastic?

$$\eta = \frac{p}{q\frac{dp}{dq}} = \frac{100e^{-0.1q}}{q(-10)e^{-0.1q}} = -\frac{10}{q}$$

$$|\eta| = \frac{10}{q} > 1 \quad \text{when} \quad \boxed{q < 10}$$

[9] *(b)* If the equilibrium price is $p_0 = 10$, find Consumers' Surplus to two decimal places.

When $p_0 = 10 = 100e^{-.1q_0} \Rightarrow e^{-.1q_0} = \frac{1}{10}$

$\Rightarrow e^{.1q_0} = 10 \Rightarrow .1q_0 = \ln 10 \Rightarrow q_0 = 10\ln 10$
≈ 23.026

$$CS = \int_0^{q_0} [D(q) - p_0]\,dq$$

$$= \int_0^{10\ln 10} [100e^{-.1q} - 10]\,dq$$

$$= \left[-1000e^{-.1q} - 10q\right]_0^{10\ln 10}$$

but $e^{-.1 \times 10\ln 10} = \frac{1}{10}$

$$= (-100 - 100\ln 10) + 1000$$

$$= \boxed{669.74}$$

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4. *[20 marks]*

Evaluate (numerical answers to two decimal places please)

[6] (a) $\int_0^1 x\sqrt{x+1}\,dx$

By substitution : Let $u = x+1 \quad du = dx$

$$\int_1^2 (u-1)\sqrt{u}\,du = \int_1^2 \left(u^{3/2} - u^{\frac{1}{2}}\right)du = \left[\frac{2}{5}u^{5/2} - \frac{2}{3}u^{3/2}\right]_1^2$$

$$= \left(\frac{2}{5}\cdot 4\sqrt{2} - \frac{2}{3}\cdot 2\sqrt{2}\right) - \left(\frac{2}{5} - \frac{2}{3}\right) \approx .6438 \approx \boxed{.64}$$

or : By parts Let $u = x \quad dv = \sqrt{x+1}\,dx$

$du = dx \quad v = \frac{2}{3}(x+1)^{3/2}$

get $$\frac{2x}{3}(x+1)^{3/2}\Big|_0^1 - \frac{2}{3}\int_0^1 (x+1)^{3/2}dx = \frac{4}{3}\sqrt{2} - \frac{2}{3}\cdot\frac{2}{5}(x+1)^{5/2}\Big|_0^1$$

$$= \frac{4}{3}\sqrt{2} + \frac{4}{15} - \frac{16}{15}\sqrt{2} = \frac{4}{15}(\sqrt{2}+1) \approx .6438 \approx \boxed{.64}$$

[6] (b) $\int_2^3 \frac{dx}{x(x-1)^2}$

$$\frac{1}{x(x-1)^2} = \frac{A}{x} + \frac{B}{x-1} + \frac{C}{(x-1)^2}$$

$$A(x-1)^2 + Bx(x-1) + Cx = 1$$

$x=1 \Rightarrow C=1$; $x=0 \Rightarrow A=1$; any other x, say $x=2$

$\Rightarrow A + 2B + 2C = 1 \Rightarrow 1 + 2B + 2 = 1 \Rightarrow 2B = -2 \Rightarrow B = -1$

$$\int_2^3 \frac{dx}{x(x-1)^2} = \int_2^3 \left[\frac{1}{x} - \frac{1}{x-1} + \frac{1}{(x-1)^2}\right]dx$$

$$= \left[\ln|x| - \ln|x-1| - \frac{1}{x-1}\right]_2^3 = \ln\left(\frac{3}{2}\right) - \ln 2 - \left(\frac{1}{2} - 1\right)$$

$$= \ln\left(\frac{3}{4}\right) + \frac{1}{2} \approx \boxed{.21}$$

Question continues on Page 12

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[8] 4. (c) Find the accumulated value of a continuous annuity after 5 years, if the payment at time t (in years) is at the rate of $1000t$ dollars per year and the annual rate of interest is 10% compounded continuously.

$$S = \int_0^T f(t)e^{r(T-t)}dt$$

$$= \int_0^5 1000te^{.10(5-t)}dt$$

$$= 1000e^{.5}\int_0^5 te^{-.10t}dt \qquad u = t \quad dv = e^{-.10t}$$

$$du = dt \qquad v = \frac{-e^{-.10t}}{.10} = -10e^{-.10t}$$

$$= 1000e^{.5}\left[-10te^{-.10t}\Big|_0^5 + 10\int_0^5 e^{-.10t}dt\right]$$

$$= 10{,}000e^{.5}\left[-5e^{-.5} - 10e^{-.10t}\Big|_0^5\right]$$

$$= 10{,}000e^{.5}\left[-5e^{-.5} + 10 - 10e^{-.5}\right]$$

$$= 10{,}000e^{.5}\left[10 - 15e^{-.5}\right]$$

$$= 10{,}000\left[10e^{.5} - 15\right]$$

$$\approx \boxed{\$\ 14{,}872.13}$$