Solved

Department of Mathematics
University of Toronto

WEDNESDAY, MARCH 5, 2003 6:10-8

MAT 133Y TERM TEST #3

Calculus and Linear Algebra for Commerce

Duration: 1 hour 50 minutes

Aids Allowed: A non-graphing calculator, with empty memory, to be supplied by student.

Instructions: Fill in the information on this page, and make sure your test booklet contains 10 pages. In addition, you should have a **multiple-choice answer sheet**, on which you should fill in your name, number, tutorial time, tutorial room, and tutor's name.

This test consists of 10 multiple choice questions, and 4 written-answer questions.
For the **multiple choice questions** you can do your rough work in the test booklet, but you must record your answer by circling the appropriate letter **on the answer sheet** with your pencil. Each correct answer is worth 4 marks; a question left blank, or an incorrect answer, or two answers for the same question is worth 0. For the **written-answer questions** , present your solutions in the space provided. The value of each written-answer question is indicated beside it. ENCLOSE YOUR FINAL ANSWER IN A BOX AND WRITE IT IN INK.

TOTAL MARKS: 100

FAMILY NAME: ________________________

GIVEN NAME: ________________________

STUDENT NO: ________________________

SIGNATURE: ________________________

TUTORIAL TIME and ROOM: ________________________

REGCODE and TIMECODE: ________________________

T.A.'S NAME: ________________________

Regcode	Timecode	Room	Regcode	Timecode	Room
T0101A	M9A	RW142	T0501C	W3C	LA341
T0101B	M9B	LM157	T0501D	W3D	NF 6
T0101C	M9C	LM123	T0601A	R4A	RW142
T0201A	M3A	LM157	T0601B	R4B	UC244
T0201B	M3B	UC85	T0601C	R4C	SS2130
T0201C	M3C	LA240	T0701A	F2A	MP118
T0201D	M3D	LA204	T0701B	F2B	SS2130
T0301A	T3A	VC212	T0701C	F2C	RW229
T0301B	T3B	NF113	T0801A	F3A	RW142
T0301C	T3C	CR403	T0801B	F3B	SS2111
T0301D	T3D	NF 7	T5101A	R5A	MP118
T0401A	W9A	LM157	T5101C	R5C	UC244
T0401B	W9B	MP118	T5201B	R6B	LM157
T0401C	W9C	LM155			
T0501A	W3A	RW143			
T0501B	W3B	RW229			

FOR MARKER ONLY	
Multiple Choice	
B1	
B2	
B3	
B4	
TOTAL	

NAME:______________________________ STUDENT NO:______________________________

PART A. Multiple Choice

1. *[4 marks]* If $f''(x) = x^2(x-1)(x-2)^3(x-3)$ then the graph of $y = f(x)$ has

A. no points of inflection

B. 1 point of inflection

C. 2 points of inflection

D. 3 points of inflection

E. 4 points of inflection

	f''	f
$(-\infty, 0)$	$-$	conc down
$(0, 1)$	$-$	conc down
$(1, 2)$	$+$	conc. up
$(2, 3)$	$-$	conc. down
$(3, \infty)$	$+$	conc up

$x = 1$ and $x = 2$ and $x = 3$ pts. of inflection

2. *[4 marks]* If $f(x) = \int_1^x \frac{dt}{t^2+1}$, then $f'(2) =$

A. $\frac{1}{5}$

B. $-\frac{3}{10}$

C. $-\frac{1}{2}$

D. $\frac{3}{10}$

E. $\frac{1}{2}$

$f'(x) = \frac{1}{x^2+1}$

$\boxed{f'(2) = \frac{1}{5}}$

NAME:_____________________________ STUDENT NO:_____________________________

3. *[4 marks]* If Newton's Method is used to approximate a root of the equation: $x^4 - 4x + 1 = 0$ by taking $x_1 = 0$, then x_3 will be closest to

A. 0.25079

B. 0.25000

C. 0.25039

D. 0.25099

E. 0.24784

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} = x_n - \frac{x_n^4 - 4x_n + 1}{4x_n^3 - 4}$$

$$x_{n+1} = \frac{3x_n^4 - 1}{4x_n^3 - 4}$$

$$x_2 = \frac{1}{4}$$

$$\boxed{x_3 = .250992063}$$

4. *[4 marks]* If the point elasticity of demand for a car is -0.8 and the price is lowered from \$20,000 to \$19,500, then we know that the number of cars sold will

A. increase by about 400 cars

B. increase by about 6250 cars

C. increase by about 2.5%

D. increase by about 2%

E. decrease by about 2%

$p = 20,000 \quad \Delta p = -500$

$$\frac{\Delta p}{p} = \frac{-500}{20,000} = -\frac{1}{40}$$

$$-0.8 = \eta = \frac{\frac{\Delta q}{q}}{\frac{\Delta p}{p}} = -40\frac{\Delta q}{q}$$

$$\boxed{\frac{\Delta q}{q} = .02}$$

NAME:____________________________ STUDENT NO:____________________________

5. *[4 marks]* $\int x^2\sqrt{1+6x^3}\,dx =$ $\frac{1}{18}\int u^{\frac{1}{2}}du$

$u = 1+6x^3$

$du = 18x^2dx$

$\frac{du}{18} = x^2dx$

$= \frac{1}{18}\cdot\frac{2}{3}u^{3/2}+K$

$= \boxed{\frac{1}{27}(1+6x^2)^{3/2}+K}$

A. $\frac{2x^3}{9}(1+6x^3)^{3/2}+K$

B. $\frac{x(1+6x^3)^{3/2}}{27}+K$

C. $\sqrt{1+6x^3}\,\ln x+K$

D. $\frac{(1+6x^3)^{3/2}}{8}+K$

E. $\frac{(1+6x^3)^{3/2}}{27}+K$

6. *[4 marks]* If $f(x) = \begin{cases} x & \text{for } x < 2 \\ 2 & \text{for } 2 \le x \end{cases}$ then $\int_{-1}^{6} f(x)\,dx =$ $-\frac{1}{2}+2+8 = 9\frac{1}{2}$

A. 10

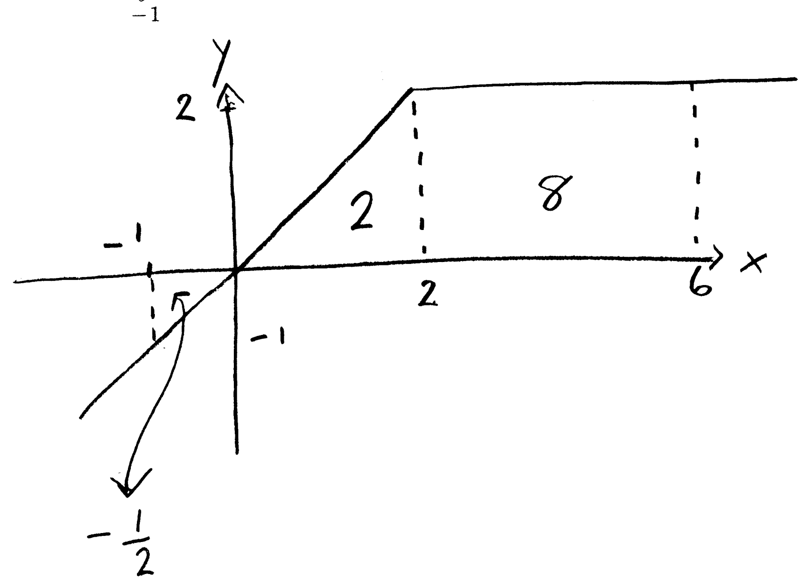
B. $11\frac{1}{2}$

C. 11

D. $9\frac{1}{2}$

E. 9

Hint: make a simple sketch.



NAME:______________________________ STUDENT NO:______________________________

7. *[4 marks]* $\displaystyle\int_0^1 \frac{e^x}{(e^x+1)^2}\,dx =$ $\displaystyle\int_2^{e+1} \frac{du}{u^2}$

$u = e^x + 1$

$du = e^x dx$

$x=0 \quad u=2$

$x=1 \quad u=e+1$

A. $\dfrac{e-1}{2(e+1)}$

B. $\dfrac{1-e}{1+e}$

C. $\dfrac{e+1}{e-1}$

D. $\dfrac{e+1}{2(e-1)}$

E. -1

$$= -\frac{1}{u}\Big|_2^{e+1}$$

$$= \frac{1}{2} - \frac{1}{e+1}$$

$$= \boxed{\frac{e-1}{2(e+1)}}$$

8. *[4 marks]* The average value of the function $f(x) = \dfrac{x+4}{x+1}$ on the interval $[1,5]$ is

A. 2

B. $4\ln 3$

C. $2\ln 2$

D. $4+3\ln 3$

E. $1+\frac{3}{4}\ln 3$

$$\frac{1}{4}\int_1^5 \frac{x+4}{x+1}\,dx$$

$$= \frac{1}{4}\int_1^5 \left(1+\frac{3}{x+1}\right) dx$$

$$= \frac{1}{4}\Big[x + 3\ln|x+1|\Big]_1^5$$

$$= \frac{1}{4}\left[4 + 3\ln 6 - 3\ln 2\right]$$

$$= \frac{1}{4}\left[4 + 3\ln\frac{6}{2}\right]$$

$$= \boxed{1 + \frac{3}{4}\ln 3}$$

NAME:________________________ STUDENT NO:________________________

9. *[4 marks]*

$$\int x^3 \ln x \, dx = \frac{x^4}{4}\ln x - \frac{1}{4}\int x^3 dx$$

$$= \boxed{\frac{x^4}{4}\ln x - \frac{1}{16}x^4 + K}$$

A. $\frac{1}{4}x^4 + x^3 \ln x + K$

B. $\frac{1}{4}x^4 \ln x - \frac{1}{16}x^4 + K$

C. $x^3 + \frac{1}{x}\ln x + K$

D. $x^3 \ln x + \frac{1}{16}x^4 + K$

E. $x^4(\ln x)^2 + x^3 \ln x + x + K$

$u = \ln x \qquad dv = x^3 dx$

$du = \frac{dx}{x} \qquad v = \frac{x^4}{4}$

10. *[4 marks]*

$$\int \frac{1}{(x-1)(x+3)}\, dx = \frac{1}{4}\int \left[\frac{1}{x-1} - \frac{1}{x+3}\right] dx$$

$$= \boxed{\frac{1}{4}\ln|x-1| - \frac{1}{4}\ln|x+3| + K}$$

A. $\frac{1}{2}\ln|x-1| + \frac{1}{2}\ln|x+3| + K$

B. $\frac{1}{4}\ln|x-1| + \frac{1}{4}\ln|x+3| + K$

C. $-\frac{1}{2}\ln|x-1| + \frac{1}{2}\ln|x+3| + K$

D. $\frac{1}{4}\ln|x-1| - \frac{1}{4}\ln|x+3| + K$

E. $-\frac{1}{4}\ln|x-1| + \frac{1}{4}\ln|x+3| + K$

$$\frac{1}{(x-1)(x+3)} = \frac{A}{x-1} + \frac{B}{x+3}$$

$A(x+3) + B(x-1) = 1$

$x = -3$: $B(-4) = 1 \qquad B = -\frac{1}{4}$

$x = 1$: $4A = 1 \qquad A = \frac{1}{4}$

NAME:_____________________________ STUDENT NO:_____________________________

PART B. Written Answer Questions

1.

[15] Let $f(x) = x^3 e^x$. Find the asymptotes, maxima/minima (relative and absolute), and points of inflection, if any; then sketch the graph of $y = f(x)$.
You may assume that $f'(x) = x^2(3+x)e^x$ and $f''(x) = x(x^2+6x+6)e^x$.

$\boxed{\text{V.A.: None}}$ $\quad \lim_{x\to\infty} x^3 e^x = \infty$

$$\lim_{x\to-\infty} \frac{x^3}{e^{-x}} = \lim_{x\to-\infty} \frac{3x^2}{-e^{-x}} = \lim_{x\to-\infty} \frac{6x}{e^{-x}} = \lim_{x\to-\infty} \frac{-6}{e^{-x}} = \lim_{x\to-\infty} -6e^x = 0$$

$\boxed{\text{H.A.: } y=0 \text{ as } x\to-\infty \text{ only.}}$

Crit. pts. $x=0, x=-3$

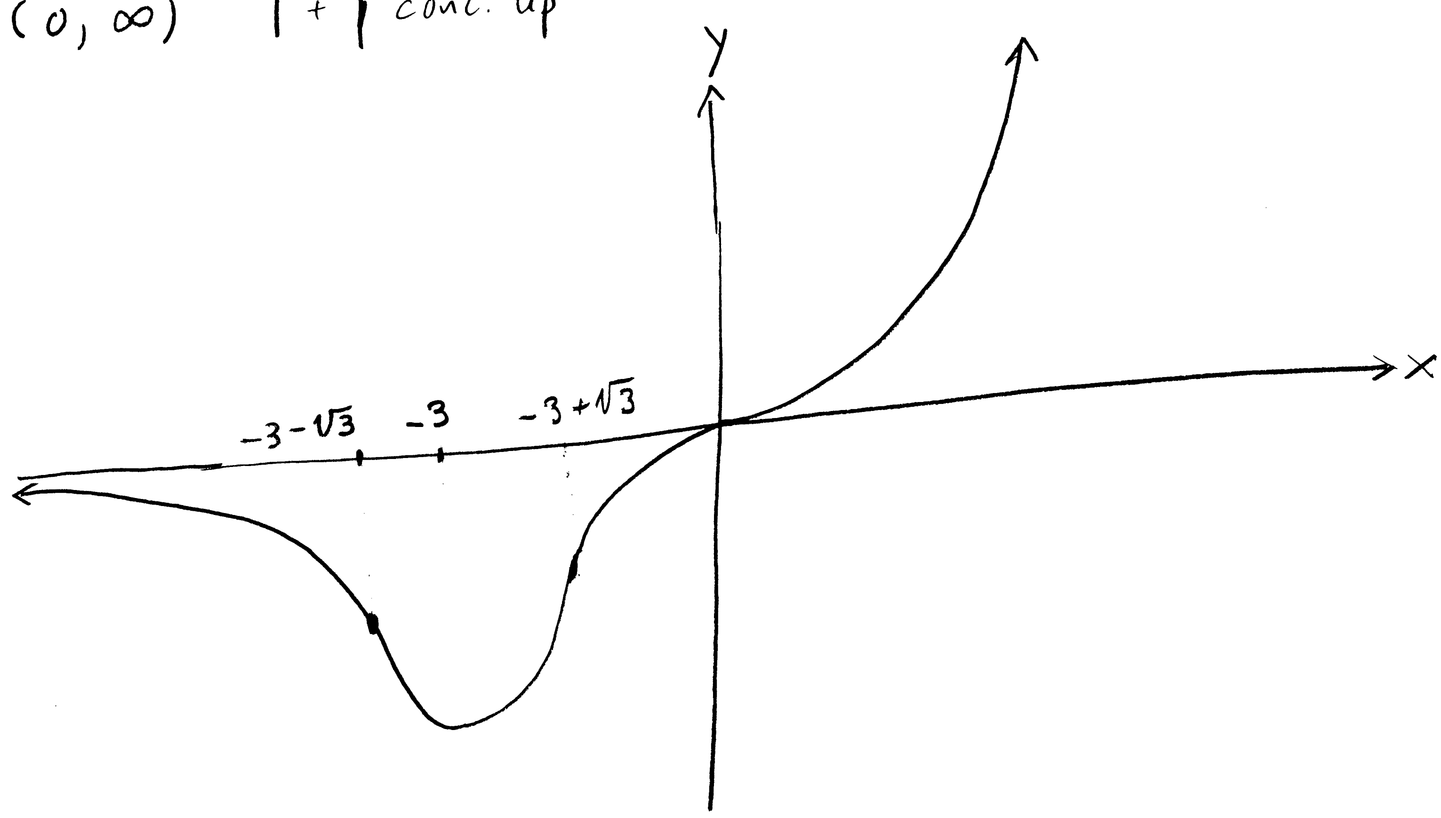
	f'	f
$(-\infty,-3)$	$-$	dec.
$(-3,0)$	$+$	inc
$(0,\infty)$	$+$	inc

$\boxed{x=-3 \quad y=-\left(\frac{3}{e}\right)^3 \text{ is a local (relative) and absolute min.}}$

$f''=0$ when $x=0$ and when $x = \dfrac{-6\pm\sqrt{12}}{2} = -3\pm\sqrt{3}$

	f''	f
$(-\infty,-3-\sqrt{3})$	$-$	conc. down
$(-3-\sqrt{3},-3+\sqrt{3})$	$+$	conc. up
$(-3+\sqrt{3}, 0)$	$-$	conc. down
$(0,\infty)$	$+$	conc. up

$\boxed{x=-3-\sqrt{3},\ -3+\sqrt{3}, \text{ and } 0 \text{ are points of inflection}}$



NAME:______________________________ STUDENT NO:______________________________

2. A monopolist's average cost function is given by:

$$\bar{c} = \frac{10 + 4\ln(2q+1)}{q}$$

The demand function for the product is known to be:

$$P = \frac{1000}{2q+1}$$

[10] a) How many items should the monopolist produce in order to maximize his profit?

[5] b) Find the point elasticity of demand when profit is maximized. What happens to the point elasticity as the number of units produced increases?

a) If π is profit, $\pi(q) = R(q) - C(q)$

$$= Pq - q\bar{c}$$

$q \geq 0$

$$\pi(q) = \frac{1000q}{2q+1} - [10 + 4\ln(2q+1)]$$

$$\frac{d\pi}{dq} = \frac{1000}{(2q+1)^2} - \frac{8}{2q+1}$$

$$= \frac{8}{(2q+1)^2}[125 - 2q - 1]$$

$$= \frac{8 \cdot 2}{(2q+1)^2}[62 - q]$$

$\frac{d\pi}{dq} = 0$ at $\boxed{q = 62}$

$\frac{d\pi}{dq} > 0$ on $[0, 62)$ and $\frac{d\pi}{dq} < 0$ on $(62, \infty)$.

So $\boxed{q = 62 \text{ is where profit is maximized}}$

b. $\eta = \dfrac{p/q}{dp/dq} = \dfrac{\frac{1000}{2q+1}}{q \cdot \frac{1000 (-1) \cdot 2}{(2q+1)^2}} = \dfrac{2q+1}{-2q} = -1 - \dfrac{1}{2q}$

always elastic.

At $q = 62$, $\boxed{\eta = -\frac{125}{124}}$

$\boxed{\text{As } q \to \infty,\ \eta \to -1}$ unit elasticity

NAME:__________ STUDENT NO:__________

3. Evalute the following limits:

[8] a) $\lim_{x\to+\infty}(x^2+e^x)^{\frac{1}{x}}$

$y=(x^2+e^x)^{\frac{1}{x}}$; $\ln y = \dfrac{\ln(x^2+e^x)}{x}$

$$\lim_{x\to\infty}\ln y = \lim_{x\to\infty}\frac{2x+e^x}{x^2+e^x} = \lim_{x\to\infty}\frac{2+e^x}{2x+e^x} \text{ by L'Hôpital}$$

$$= \lim_{x\to\infty}\frac{e^x}{2+e^x} = \lim_{x\to\infty}\frac{e^x}{e^x} = 1$$

$$\ln y \to 1 \Rightarrow y \to e^1 = \boxed{e}$$

Alternatively: $\displaystyle\lim_{x\to\infty}\frac{2x+e^x}{x^2+e^x} = \lim_{x\to\infty}\frac{2xe^{-x}+1}{x^2e^{-x}+1} = 1$

since xe^{-x} and $x^2e^{-x}\to 0$ as $x\to\infty$.

Again, $y\to\boxed{e}$

[7] b) $\displaystyle\lim_{x\to2}\frac{x-\ln(x-1)-2}{(x-2)^2} = \lim_{x\to2}\frac{1-\frac{1}{x-1}}{2(x-2)}$ still $\frac{0}{0}$

$$= \lim_{x\to2}\frac{\frac{1}{(x-1)^2}}{2} = \boxed{\frac{1}{2}}$$

Alternatively, $\displaystyle\frac{1-\frac{1}{x-1}}{2(x-2)} = \frac{x-2}{2(x-1)(x-2)} = \frac{1}{2(x-1)}$

$\to \boxed{\frac{1}{2}}$ as $x\to2$

NAME:______________________________ STUDENT NO:______________________________

4. A certain commodity has the demand curve

$$(p+1)(q+1) = 6$$

($p =$ unit price, $q =$ quantity bought per month) and the supply curve

$$q = 2p$$

($p =$ unit price, $q =$ quantity produced per month).

[3] a) Find the equilibrium price and quantity bought/produced per month.

[12] b) Find the **consumers'** surplus for this commodity.

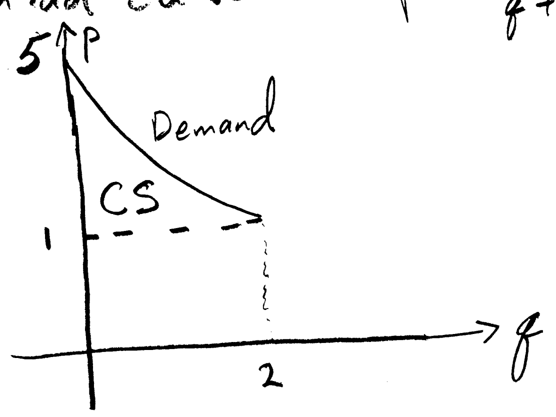
a) To find the intersection of the two curves, set $q = 2p$

$(p+1)(2p+1) = 6$ $\qquad$ $2p^2 + 3p - 5 = 0$

$\qquad\qquad\qquad\qquad\quad$ $(2p+5)(p-1) = 0$

$\boxed{\text{Equilibrium is at } p = 1,\ q = 2}$

b) Demand curve is $p = \dfrac{6}{q+1} - 1$



$$CS = \int_0^2 \left[\frac{6}{q+1} - 2\right] dq = \Big[6 \ln|q+1| - 2q\Big]_0^2$$

$$\boxed{CS = 6 \ln 3 - 4}$$

Alternatively $q = \dfrac{6}{p+1} - 1$ and

$$CS = \int_1^5 \left[\frac{6}{p+1} - 1\right] dp = \Big[6 \ln|p+1| - p\Big]_1^5$$

$$\boxed{CS = 6 \ln 3 - 4}$$