Solved

Department of Mathematics
University of Toronto

WEDNESDAY, MARCH 6, 2002, 6:10 - 8:00 PM
MAT 133Y TERM TEST #3

Calculus and Linear Algebra for Commerce
Duration: 1 hour 50 minutes

Aids Allowed: A non-graphing calculator, with empty memory, to be supplied by student.

Instructions: Fill in the information on this page, and make sure your test booklet contains 10 pages. In addition, you should have a **multiple-choice answer sheet**, on which you should fill in your name, number, tutorial time, tutorial room, and tutor's name.

This test consists of 10 multiple choice questions, and 4 written-answer questions.
For the **multiple choice questions** you can do your rough work in the test booklet, but you must record your answer by circling the appropriate letter **on the answer sheet** with your pencil. Each correct answer is worth 4 marks; a question left blank, or an incorrect answer, or two answers for the same question is worth 0. For the **written-answer questions** , present your solutions in the space provided. The value of each written-answer question is indicated beside it.
ENCLOSE YOUR FINAL ANSWER IN A BOX AND WRITE IT IN INK.

TOTAL MARKS: 100

FAMILY NAME: ______________________

GIVEN NAME: ______________________

STUDENT NO: ______________________

SIGNATURE: ______________________

TUTORIAL TIME and ROOM: ______________________

REGCODE and TIMECODE: ______________________

T.A.'S NAME: ______________________

Regcode	Timecode	Room	Regcode	Timecode	Room
T0101A	M9A	SS2111	T0501C	W3C	RW143
T0101B	M9B	SS2128	T0501D	W3D	SS2128
T0101C	M9C	SS2130	T0601A	R4A	UC52
T0201A	M3A	LA341	T0601B	R4B	UC85
T0201B	M3B	RW229	T0601C	R4C	UC144
T0201C	M3C	SS2111	T0701A	F2A	LM158
T0201D	M3D	LA240	T0701B	F2B	WI523
T0301A	T3A	SS2128	T0701C	F2C	SS1072
T0301B	T3B	MP137	T0801A	F3A	SS2130
T0301C	T3C	TF203	T0801B	F3B	SS2111
P0301D	T3D	NF332	T5101A	R5A	UC52
T0401A	W9A	SS1074	T5101B	R5B	UC85
T0401B	W9B	SS2111	T5101C	R5C	UC244
T0401C	W9C	LM123	T5201A	R6B	UC144
T0501A	W3A	LM123	T5201B	R6B	UC244
T0501B	W3B	LM157			

FOR MARKER ONLY	
Multiple Choice	
B1	
B2	
B3	
B4	
TOTAL	

PART A. Multiple Choice

1. *[4 marks]*
   The slope of the tangent to the curve:

$$x \ln y + 5^x = 5y$$

$$\ln y + \frac{x}{y} y' + 5^x \ln 5 = 5y'$$

at the point (1,1) is

at (1,1) $\ln 1 + y' + 5 \ln 5 = 5y'$

$5 \ln 5 = 4y'$

$y' = \boxed{\frac{5 \ln 5}{4}}$

A. $\dfrac{1 + 5\ln 5}{5}$

B. $\dfrac{1}{4}$

C. $\dfrac{5\ln 5}{4}$

D. $\dfrac{5}{4}$

E. $\dfrac{5}{4\ln 5}$

2. *[4 marks]*
   The graph of $f(x) = \dfrac{2x^2 - 3x - 2}{x^2 - x - 2}$ has

A. a vertical asymptote at $x = -1$ and a horizontal asymptote at $y = 2$

B. a vertical asymptote at $x = 2$ and a horizontal asymptote at $y = 2$

C. a vertical asymptote at $x = -1$ and no horizontal asymptote

D. vertical asymptotes at $x = -1$ and $x = 2$ and a horizontal asymptote at $y = 2$

E. vertical asymptotes at $x = -1$ and $x = 2$ and no horizontal asymptote

$f(x) = \dfrac{(2x+1)(x-2)}{(x+1)(x-2)} = \dfrac{2x+1}{x+1}$ when $x \neq 2$

$\boxed{\text{V.A at } x=-1 \text{ only}}$ so not B, D, E

$\lim_{x \to \pm\infty} f(x) = \lim_{x \to \pm\infty} \dfrac{2 + \frac{1}{x}}{1 + \frac{1}{x}} = 2$

$\boxed{\text{H.A at } y=2}$

3. *[4 marks]*
The graph of $f(x) = e^x + e^{-x}$ is

A. increasing when $x > 0$ and always concave upward

B. increasing when $x < 0$ and always concave upward

C. increasing and concave upward everywhere

D. increasing everywhere and concave upward when $x > 0$

E. increasing and concave downward when $x > 0$

$f'(x) = e^x - e^{-x} = e^{-x}(e^{2x} - 1) > 0$ only when $x > 0$.
so not B, C, D

$f''(x) = e^x + e^{-x} > 0$ everywhere so concave upward

4. *[4 marks]* $\frac{0}{0}$

$$\lim_{x\to 0} \frac{\sqrt{1+x} - 1}{e^{3x} - 1} = \lim_{x\to 0} \frac{\frac{1}{2\sqrt{1+x}}}{3e^{3x}} = \frac{\frac{1}{2}}{3} = \frac{1}{6}$$

A. $\frac{1}{3}$

B. $\frac{1}{12}$

C. $\frac{1}{2}$

D. $\frac{1}{6}$

E. 1

5. *[4 marks]*

$\lim_{x\to 1} x^{\left(\frac{2}{x-1}\right)} =$

A. 1

B. ∞

C. e^2

D. 2

E. e

$y = x^{\frac{2}{x-1}}$

$\ln y = \frac{2\ln x}{x-1}$ $\quad \frac{0}{0}$

$\lim_{x\to 1} \ln y = \lim_{x\to 1} \frac{2\ln x}{x-1} = \lim_{x\to 1} \frac{\frac{2}{x}}{1} = 2$

$\lim_{x\to 1} y = \lim_{x\to 1} e^{\ln y} = e^2$

6. *[4 marks]*

Given the demand function

$$q = p^{10}e^{-1.1(p+1)}$$

the point of elasticity of demand when $p = 10$ is

A. -1.21

B. -1.1

C. $-.909$

D. 0

E. -1

$\mu = \frac{\frac{dq}{q}}{\frac{dp}{p}} = \frac{p}{q}\frac{dq}{dp}$

$\frac{dq}{dp} = 10p^9 e^{-1.1(p+1)} - 1.1\, p^{10} e^{-1.1(p+1)}$

$\frac{p}{q}\frac{dq}{dp} = \frac{\cancel{e^{-1.1(p+1)}}\left[10p^{10} - 1.1\, p^{11}\right]}{p^{10}\cancel{e^{-1.1(p+1)}}}$

$\mu = 10 - 1.1p$

when $p = 10$, $\mu = 10 - 11 = -1$

7. *[4 marks]*
If Newton's Method is used to approximate a solution to the equation $e^x - 4x = 0$, starting with $x_1 = 3$, then x_2 is closest to

A. 8.09

B. 0.12

C. 2.16

D. 2.23

E. 2.50

$$X_2 = X_1 - \frac{f(x_1)}{f'(x_1)}$$

$$X_2 = X_1 - \frac{e^{x_1} - 4x_1}{e^{x_1} - 4} \quad . \quad \text{If } X_1 = 3,$$

$$X_2 = 3 - \frac{e^3 - 12}{e^3 - 4} \approx 2.497$$

8. *[4 marks]*
$\int_0^2 \frac{x^2}{x-3}\, dx =$

A. $8 - 9\ln 3$

B. $9 - 6\ln 3$

C. $6 + 3\ln 3$

D. $8 + 6\ln 3$

E. $6 - 9\ln 3$

$$\begin{array}{r|l} & x+3 \\ x-3 & x^2 \\ & x^2 - 3x \\ \hline & 3x \\ & 3x - 9 \\ \hline & 9 \end{array}$$

$$\int_0^2 \frac{x^2}{x-3}\, dx = \int_0^2 \left[(x+3) + \frac{9}{x-3}\right] dx$$

$$= \left[\frac{x^2}{2} + 3x + 9\ln|x-3|\right]_0^2$$

$$= (2+6) - 9\ln 3$$

9. *[4 marks]*

If $f(x) = \int_1^x \sqrt{t^2+5}\,dt$, then $f'(2) =$

A. 2

B. 3

C. $\frac{2}{3}$

D. $3-\sqrt{6}$

E. 1

$f'(x) = \sqrt{x^2+5}$

$f'(2) = \sqrt{9} = 3$

10. *[4 marks]*

The area of the region in the xy plane above $y = 1$ and below $y = e^x$ from $x = 2$ to $x = 3$ is

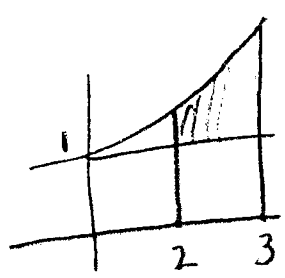
A. $\frac{e^3}{3} - \frac{e^2}{2} - 1$

B. $e^3 - e^2 - 1$

C. $e^3 - e^2$

D. $3e^3 - 2e + 1$

E. $\ln 6$



$$\text{Area} = \int_2^3 (e^x - 1)\,dx$$

$$= \left[e^x - x\right]_2^3$$

$$= (e^3 - 3) - (e^2 - 2)$$

$$= e^3 - e^2 - 1$$

PART B. Written-Answer Questions

1. *[15 marks]*
Graph $y = f(x)$ where
$$f(x) = \frac{x+1}{\sqrt{x}}$$
Find all intervals where f is increasing, decreasing, concave upward and concave downward.
Find and label all asymptotes, maximum, minimum and inflection points.

$f(x) = \sqrt{x} + \frac{1}{\sqrt{x}}$ defined only when $x > 0$.

$\boxed{\text{V.A. at } x=0}$ $\quad \lim_{x\to\infty} f(x) = \infty$ so $\boxed{\text{no H.A}}$

$f'(x) = \frac{1}{2}x^{-\frac{1}{2}} - \frac{1}{2}x^{-3/2} = \frac{1}{2x^{3/2}}(x-1)$

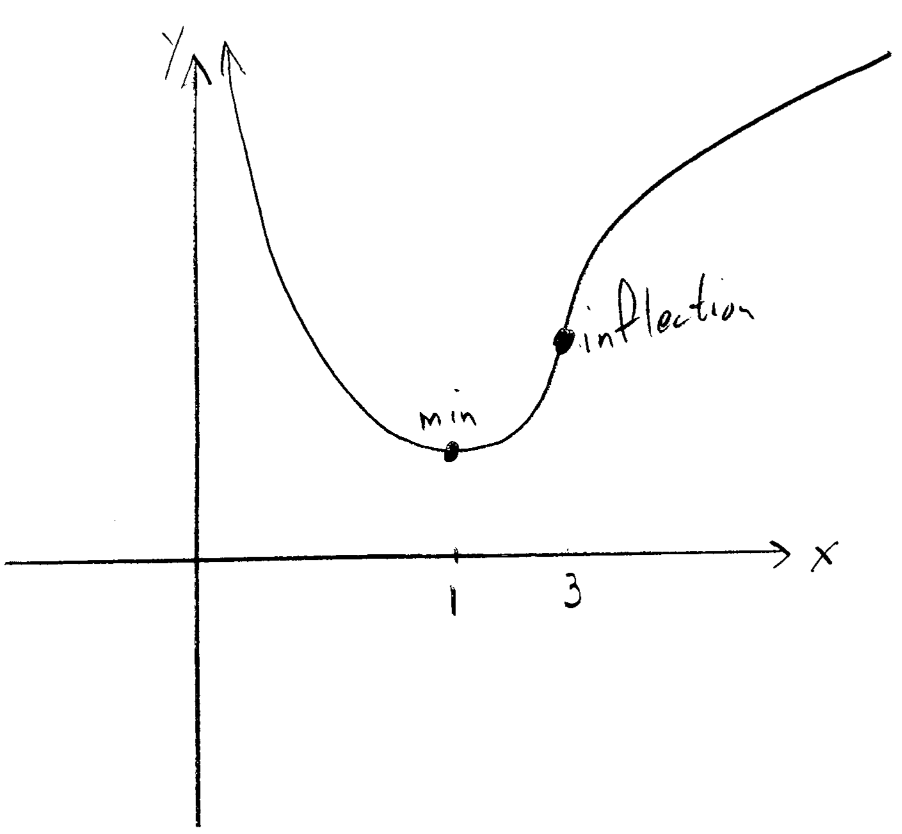
$f''(x) = -\frac{1}{4}x^{-\frac{3}{2}} + \frac{3}{4}x^{-\frac{5}{2}} = \frac{1}{4x^{5/2}}(3-x)$

$f' < 0$ on $(0,1)$ $\quad f' > 0$ on $(1,\infty)$

f is decreasing on $(0,1)$; f is increasing on $(1,\infty)$
f has an absolute minimum at $x=1$

$f'' > 0$ on $(0,3)$ $\quad f'' < 0$ on $(3,\infty)$

f is concave up on $(0,3)$; f is concave down on $(3,\infty)$
f has a pt. of inflection at $x=3$



2. *[15 marks]*

The price (V) of a parcel of land bought for speculation is estimated to increase according to the formula $V(t) = 2000e^{(t^{\frac{1}{4}})}$, where t is in years and V is in dollars. If the interest rate is expected to remain 10% per year, compounded continuously, how long should you be planning to wait until selling, assuming that you wish to maximize present value?

Let $A(t)$ be the present value if you wait t years.

Then
$$A(t) = V(t)e^{-.10t} = 2000e^{t^{\frac{1}{4}} - .10t}$$

$$\frac{dA}{dt} = 2000e^{t^{\frac{1}{4}} - .10t}\left[\frac{1}{4}t^{-3/4} - .10\right]$$

$$\frac{dA}{dt} = 0 \text{ when } t^{-3/4} = .4$$

$$t = (.4)^{-4/3} \approx \boxed{3.93 \text{ years}}$$

$t \in [0, \infty)$ and $\frac{dA}{dt}$ is cont. on $(0, \infty)$

When t is very near 0 $t^{-3/4}$ is BIG so $\frac{dA}{dt} > 0$ on $(0, 3.93)$

When t is BIG, $t^{-3/4}$ is near zero so $\frac{dA}{dt} < 0$ on $(3.93, \infty)$.

Hence $t = 3.93$ is where the global max is.

3. *[15 marks]*
When a certain factory produces q units, its marginal cost of production is $e^{-q}\sqrt{e^{-q}+3}$.
Find the total cost of producing q units if the start-up cost (cost when $q = 0$) is 4.

$$\frac{dc}{dq} = e^{-q}\sqrt{e^{-q}+3}$$

$$C(q) = \int e^{-q}\sqrt{e^{-q}+3}\,dq$$

Let $u = e^{-q}+3$ $\quad du = -e^{-q}dq$

$$C(q) = -\int u^{\frac{1}{2}}du = -\frac{2}{3}u^{3/2} + K$$

$$C(q) = K - \frac{2}{3}(e^{-q}+3)^{3/2}$$ but $C(0) = 4$

$$4 = K - \frac{2}{3}(1+3)^{3/2}$$

$$4 = K - \frac{2}{3}\cdot 8 \qquad K = \frac{28}{3}$$

$$\boxed{C(q) = \frac{28}{3} - \frac{2}{3}(e^{-q}+3)^{3/2}}$$

4. *[15 marks]*
A monopoly has a total cost function $c(q) = 1000 + 120q - 6q^2$ for its product, which has a marginal revenue function $\frac{dr}{dq} = 372 - 6q - 6q^2$. Find the consumers' surplus at the point where the monopoly has maximum profit. You may assume revenue is zero when $q = 0$.

Max profit is when $\frac{dr}{dq} = \frac{dc}{dq}$

$$372 - 6q - 6q^2 = 120 - 12q$$
$$252 + 6q - 6q^2 = 0$$
$$q^2 - q - 42 = 0$$
$$(q-7)(q+6) = 0 \quad \text{So when } \boxed{q_0 = 7}$$

To find the demand function:

$$r(q) = 372q - 3q^2 - 2q^3 + K$$ but $r=0$ when $q=0$, so $K=0$.

$$pq = r(q) = 372q - 3q^2 - 2q^3$$

$p = 372 - 3q - 2q^2$ is the demand function

and when $q = 7$, $p = 372 - 21 - 98$ $\boxed{p_0 = 253}$

$$CS = \int_0^{q_0} [D(q) - p_0]\, dq$$
$$= \int_0^7 [(372 - 3q - 2q^2) - 253]\, dq$$
$$= \int_0^7 [119 - 3q - 2q^2]\, dq$$
$$= \left[119q - \frac{3q^2}{2} - \frac{2q^3}{3}\right]_0^7$$
$$= 119 \cdot 7 - 3 \cdot \frac{49}{2} - \frac{2 \cdot 7^3}{3}$$
$$\approx \boxed{530.83}$$