Part 2: Long Answers (60 marks)

Show your work for full marks

2. Evaluate the following limits.

   (a) (10 points)

$$\lim_{x\to\infty} \frac{(2-x)(2+3x)}{6x+x^2}$$

$$= \lim_{x\to\infty} \frac{-3x^2+4x+4}{x^2+6x}$$

$$= \lim_{x\to\infty} \frac{x^2\left(-3+\frac{4}{x}+\frac{4}{x^2}\right)}{x^2\left(1+\frac{6}{x}\right)}$$

$$= \lim_{x\to\infty} \frac{-3+\frac{4}{x}+\frac{4}{x^2}}{1+\frac{6}{x}}$$

$$= \frac{-3+0+0}{1+0}$$

$$= -3.$$

(b) (5 points)

$$\lim_{x\to 0} x(\ln(x))^2$$

Limit DNE as ln only defined on $(0,\infty)$.
To calculate the right sided limit, use L'Hopital's Rule twice:

$$\lim_{x\to 0^+} x(\ln x)^2 = \lim_{x\to 0^+} \frac{(\ln x)^2}{\left(\frac{1}{x}\right)} \overset{L'H}{=} \lim_{x\to 0^+} \frac{\left(\frac{2\ln x}{x}\right)}{\left(\frac{-1}{x^2}\right)}$$

$$= \lim_{x\to 0^+} -2x\ln x = \lim_{x\to 0^+} -2\cdot\frac{\ln x}{\left(\frac{1}{x}\right)}$$

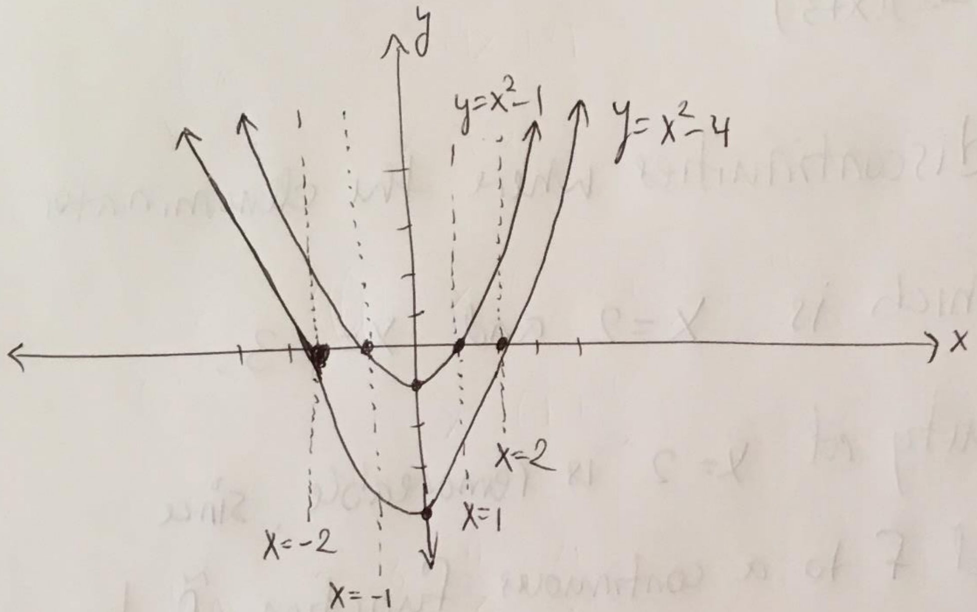
$$\overset{L'H}{=} \lim_{x\to 0^+} -2\cdot\frac{\left(\frac{1}{x}\right)}{\left(\frac{-1}{x^2}\right)} = \lim_{x\to 0^+} -2(-x)$$

$$= 0.$$

3. (a) (8 points) Solve the following inequality for x.

$$\frac{x^2-1}{x^2-4} < 0$$

Graph $y = x^2 - 1$ and $y = x^2 - 4$:



The inequality holds for all x such that the two parabolas have different sign, which is precisely the intervals $(-2,-1) \cup (1,2)$.

(b) (7 points) Find the value(s) of x for which $f(x) = \frac{x-2}{x^2+x-6}$ is discontinuous. Justify your answer.

Factor the denominator:

$$f(x) = \frac{x-2}{(x-2)(x+3)}$$

So, f has discontinuities where the denominator vanishes, which is $x=2$ and $x=-3$.

The discontinuity at $x=2$ is removeable, since we can extend f to a continuous function $\tilde{f}$ by setting $\tilde{f}(2) = \lim_{x\to 2} f(x) = \lim_{x\to 2} \frac{\cancel{x-2}}{\cancel{(x-2)}(x+3)} = \frac{1}{5}$.

Marking scheme for questions 2 and 3

2) a) 10 marks for correct answer with justification, 8 marks for correct answer with no justification (or wrong justification), 8 marks for mostly correct justification but wrong final answer, 5 points for expanding the numerator and stopping there.

b) Almost nobody noticed that the limit is only defined on one side, so I didn't take off marks for that. 5 points for (properly) using L'Hopital twice, minus a mark or two for incorrect algebraic manipulations (for example, a lot of them said that $(\log x)^2 = 2 \log x$). 0 points if they clearly didn't know what they were doing.

3) a) 8 point for correctly finding the proper intervals, 6 points if they made a chart (or tried to make a chart) of where each factor was + and - but didn't properly analyze it, minus a mark or two for algebraic errors.

b) 7 points for correctly identifying the discontinuities at $x=2,-3$. Minus 1 point for failing to note that the discontinuity at $x=2$ is removable (which pretty much everybody did).

4. (15 points) Let $p = \frac{q^2}{5} + \frac{60(q+3)^{1/3}}{q}$ and $q = \sqrt{m^2+16}$. Let $r = pq$. Determine $\frac{dr}{dm}$ when $m = 3$.

$q(m) = \sqrt{m^2+16} \Rightarrow q(3) = 5$ 1

$\frac{dq(m)}{dm} = \frac{d(\sqrt{m^2+16})}{dm} = \frac{2m}{2\sqrt{m^2+16}}$ 2 $\Rightarrow \frac{dq}{dm}, q'(m) = \frac{m}{\sqrt{m^2+16}}$

$p(5) = 5 + 24 = 29$ 1 $\Rightarrow \frac{dq}{dm}\Big|_{m=3} = \frac{3}{5}$ 1

$\frac{dp}{dq} = \frac{d}{dq}\left(\frac{q^2}{5} + \frac{60(q+3)^{1/3}}{q}\right) =$

3

$= \frac{2q}{5} + \frac{20q(q+3)^{-2/3} - 60(q+3)^{1/3}}{q^2}$

$\frac{dp}{dq}\Big|_{q=5 \text{ or } m=3} = \frac{2 \cdot 5}{5} + \frac{100 \cdot 8^{-2/3} - 60 \cdot 8^{1/3}}{25} = 2 + \left(\frac{-19}{5}\right) = \frac{-9}{5}$ 2

$\frac{dr}{dq} = \frac{d(qp)}{dq} = p + q\frac{dp}{dq}$ 3

$\Rightarrow \frac{dr}{dq}\Big|_{q=5} = p + q\frac{dp}{dq}\Big|_{q=5} = 29 + 5 \cdot \left(-\frac{9}{5}\right) = 20$

2

$\frac{dr}{dm}\Big|_{m=3 \Rightarrow q=5} = \frac{dr}{dq}\frac{dq}{dm}\Big|_{m=3 \Rightarrow q=5} = 20 \cdot \frac{3}{5} = 12$

5. (15 points) A particle is travelling with position $f(t)$ at time $t \geq 0$. If

$$f(t) = (t+1)e^{\frac{1}{2}t^2 - t},$$

find the acceleration of the particle. (Hint: find the velocity $v(t) = f'(t)$ and then the acceleration $a(t) = v'(t)$. You will need the product rule and the chain rule.)

7 $\frac{df(t)}{dt} = (t+1)(t-1)e^{\frac{1}{2}t^2-t} + e^{\frac{1}{2}t^2-t} = t^2 e^{\frac{1}{2}t^2-t}$ 2 1 2 2

7 $\frac{d^2 f(t)}{dt^2} = \frac{d}{dt}\left(t^2 e^{\frac{1}{2}t^2-t}\right)$

$= 2te^{\frac{1}{2}t^2-t} + t^2(t-1)e^{\frac{1}{2}t^2-t}$ 2 2

$= t(t^2-t+2)e^{\frac{1}{2}t^2-t}$ 2

+1 fin

Faculty of Arts and Science
University of Toronto

MAT133Y Term Test 2

Friday June 22, 2018, 9:10 am – 11:00 am
Duration - 110 minutes

Surname: ______________________

Given Name: ______________________

Student Number: ______________________

Tutorial Section: ______________________

Multiple Choice Answer Sheet

Question Number	Answer
1	A
2	C
3	E
4	E
5	E
6	C
7	A
8	D
9	B
10	B