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Department of Mathematics
University of Toronto

Tuesday, Jan. 16, 2018, 6:10-8:00 PM

MAT 133Y TERM TEST #2

Calculus and Linear Algebra for Commerce
Duration: 1 hour 50 minutes

Aids Allowed: A TI-30X IIS calculator, to be supplied by student. **No other calculator is permitted.**

Instructions: Fill in the information on this page, and make sure your test booklet contains 10 pages. In addition, you should have a **multiple-choice answer sheet**, on which you should fill in your name, number, tutorial time, tutorial room, and tutor's name.

This test consists of 10 multiple choice questions, and 4 written-answer questions.
For the **multiple choice questions** you can do your rough work in the test booklet, but you must record your answer by circling the appropriate letter **on the answer sheet** with your pencil. Each correct answer is worth 4 marks; a question left blank, or an incorrect answer, or two answers for the same question is worth 0. For the **written-answer questions**, present your solutions in the space provided. The value of each written-answer question is indicated beside it.

ENCLOSE YOUR FINAL ANSWER IN A BOX AND WRITE IT IN INK.

TOTAL MARKS: 100

FAMILY NAME: ______________________

GIVEN NAME: ______________________

STUDENT NO: ______________________

SIGNATURE: ______________________

TUTORIAL TIME and ROOM: ______________________

REGCODE and TIMECODE: ______________________

T.A.'S NAME: ______________________

Regcode	Timecode	Room	**Regcode**	Timecode	Room
T0101	M9A	RS310	T0502	W3B	GB120
T0102	M9B	BA2135	T0503	W3C	BA2145
T0103	M9C	HA316	T0601	R4A	BA2195
T0104	M9D	LM157	T0602	R4B	SS2106
T0201	M3A	UC52	T0603	R4C	BA1240
T0202	M3B	UC87	T0604	R4D	SS2105
T0203	M3C	BA3012	T0701	F2A	BA2135
T0204	M3D	SS2127	T0702	F2B	SS2105
T0301	T3A	MS3278	T0703	F2C	BA2165
T0302	T3B	UC144	T0801	F3A	ES4000
T0303	T3C	BA1220	T0802	F3B	SS2105
T0304	T3D	BA3012	T0803	F3C	RW143
T0401	W9A	AB107	T5101	M5A	MS4171
T0402	W9B	BA2185	T5102	M5B	BA2175
T0403	W9C	LM157	T5103	M5C	BA2139
T0501	W3A	BA3116			

FOR MARKER ONLY	
Multiple Choice	
B1	
B2	
B3	
B4	
TOTAL	

NAME: ____________________ STUDENT NO: ____________

PART A. Multiple Choice

1. *[4 marks]*

If $f(x) = \dfrac{x^2 - 5x + 2}{3x + 2}$, then $f'(0) =$

$$f'(x) = \frac{(3x+2)(2x-5) - (x^2-5x+2)\cdot 3}{(3x+2)^2}$$

$$f'(0) = \frac{2\cdot(-5) - 2\cdot 3}{4} = -\frac{16}{4} = -4 \quad \text{A}$$

A. -4

B. 4

C. 2

D. 1

E. -1

2. *[4 marks]*

If $y = \dfrac{1}{\sqrt[3]{5x^2 - 3}}$, then $\dfrac{dy}{dx} =$

$$y = (5x^2-3)^{-\frac{1}{3}}$$

$$\frac{dy}{dx} = -\frac{1}{3}(5x^2-3)^{-\frac{4}{3}}\cdot 10x$$

$$= \frac{-10x}{3\sqrt[3]{(5x^2-3)^4}} \quad \text{D}$$

A. $\dfrac{1}{3(5x^2-3)\sqrt[3]{5x^2-3}}$

B. $\dfrac{10x}{3\sqrt[3]{(5x^2-3)^2}}$

C. $\dfrac{1}{\sqrt[3]{10x-3}}$

D. $\dfrac{-10x}{3\sqrt[3]{(5x^2-3)^4}}$

E. $\dfrac{-10x^2}{3\sqrt[3]{(5x^2-3)^4}}$

NAME: ____________________ STUDENT NO: ____________

3. *[4 marks]*

Let $y = (f(x))^3$, $f'(2) = 3$, $f(2) = 6$. Find y' when $x = 2$.

A. 282

B. 324

C. 140

D. 220

E. 186

$$y' = 3[f(x)]^2 \cdot f'(x)$$
$$= 3[f(2)]^2 \cdot f'(2)$$
$$= 3 \cdot 6^2 \cdot 3$$
$$= 324 \quad \text{B}$$

4. *[4 marks]*

Let $f(x) = \begin{cases} x-3 & \text{if } x < 0 \\ 3 & \text{if } 0 \le x < 1 \\ \text{undefined} & \text{if } x = 1 \\ 3 & \text{if } 1 < x \le 2 \\ x+1 & \text{if } 2 < x \end{cases}$.

$f(x)$ is not continuous at

A. $x = 2$ and $x = 1$ only

B. $x = -3$ and $x = 0$ only

C. $x = 0$ and $x = 1$ only

D. $x = 0$ and $x = 1$ and $x = 2$

E. $x = 1$ and $x = 3$ only

$x \to 0^-$ $f \to -3$ $x \to 0^+$ $f \to 3$ not cont.

$\longrightarrow$ not cont because f not defined at $x=1$

$x \to 2^-$ $f \to 3$ cont at $x = 2$
$x \to 2^+$ $f \to 3$

Cont everywhere else.
so C

NAME: ____________________ STUDENT NO: ____________

5. *[4 marks]*

$f(x) = 3x - x^2 + |2x+3|$ does **not** have a derivative at $x =$

A. $\frac{3}{2}$

B. $\frac{2}{3}$

C. $-\frac{3}{2}$

D. 0

E. 4

The only trouble is at $|\,\,|$, i.e. $2x+3=0$ and $x = -\frac{3}{2}$ C

6. *[4 marks]*

Let $f(x) = \ln(2x+1)$. Then $f'''(0) =$

A. ~~0~~ 0

B. ~~2~~ 4

C. ~~4~~ 8

D. ~~-8~~ -16

E. ~~8~~ 16

$f'(x) = \frac{2}{2x+1}$

$f''(x) = \frac{-4}{(2x+1)^2}$

$f'''(x) = \frac{16}{(2x+1)^3}$

$f'''(0) = 16$ E

NAME: ____________________ STUDENT NO: ____________

7. *[4 marks]*

If $f(x) = \dfrac{2e^{3x}\sqrt{x^2+1}(x^2+x+1)}{(x+1)^6}$, then $f'(0) =$

A. -4

B. 4

C. -2

D. 2

E. 1

$\ln f = \ln 2 + 3x + \frac{1}{2}\ln(x^2+1) + \ln(x^2+x+1) - 6\ln(x+1)$

$\frac{1}{f} f' = 3 + \frac{x}{x^2+1} + \frac{2x+1}{x^2+x+1} - \frac{6}{x+1}$

$\frac{1}{f(0)} f'(0) = 3 + 0 + 1 - 6 = -2$

$f'(0) = -2f(0)$ but $f(0) = 2$

$f'(0) = -4$ A

8. *[4 marks]*

Given that $\dfrac{d}{dx}(c^x - x^c) = 0$ when $x = 1$, and that c is a constant greater than 0, then $c =$

A. e^2

B. 1

C. 2

D. e

E. e^{-1}

$\frac{d}{dx} c^x = c^x \ln c$

$\frac{d}{dx} x^c = cx^{c-1}$

So $c^x \ln c = cx^{c-1}$ when $x = 1$,

i.e. $c \ln c = c$ $\quad c > 0$

so $\ln c = 1$

$c = e$ D

NAME: ____________________ STUDENT NO: ____________

9. *[4 marks]*

If $y = (7x)^{(2x)}$, then $\dfrac{dy}{dx} =$

$\ln y = 2x \ln(7x)$

$\frac{1}{y} y' = 2\ln(7x) + 2x \cdot \frac{1}{7x} \cdot 7$

$= 2\ln(7x) + 2$

$y' = 2(\ln(7x) + 1)y$ D

A. $(2x)(7x)^{(2x-1)}$

B. $(7x)^{(2x)} \ln(7x)$

C. $(7x)^{(2x)} \ln(7)$

D. $2(\ln(7x) + 1)(7x)^{(2x)}$

E. $\left(2\ln(7x) + \dfrac{2}{x}\right)(7x)^{(2x)}$

10. *[4 marks]*

If a country's savings, S, and national income, I, are related as follows:

$$2S^2 + I^2 = 3SI$$

then when $S = 2$ and $I = 4$, the marginal propensity to save is

$4S\frac{dS}{dI} + 2I = 3\frac{dS}{dI}I + 3S$

$8\frac{dS}{dI} + 8 = 12\frac{dS}{dI} + 6$

$2 = 4\frac{dS}{dI}$

$\frac{1}{2} = \frac{dS}{dI}$ D

A. $\dfrac{3}{4}$

B. $-\dfrac{8}{5}$

C. $\dfrac{5}{6}$

D. $\dfrac{1}{2}$

E. 2

A

NAME: ______________________ STUDENT NO: ______________

PART B. Written-Answer Questions

1. *[16 marks]*

[8] (a) Solve the inequality $\frac{e^x \ln x}{x-4} \geq 0$.

$f(x) = \frac{e^x \ln x}{x-4}$

f is undefined for $x \leq 0$, and $x = 4$. $f = 0$ at $x = 1$.

f is cont and not $= 0$ on each of the open intervals in the table.

	x	f
$(0,1)$	$\frac{1}{2}$	$+$
$(1,4)$	3 (or 2)	$-$
$(4,\infty)$	10^{100}	$+$

$e^x > 0$ for every x.

At $x = \frac{1}{2}$ $\ln x < 0$ and $x - 4 < 0$

At $x = 2$ $\ln x > 0$ but $x - 4 < 0$

At $x > 4$ everything is > 0

$\boxed{(0,1] \cup (4,\infty)}$

$\boxed{0 < x \leq 1 \text{ or } x > 4}$

[8] (b) Using Newton's Method to approximate the solution of $x^4 - 2x - 3 = 0$ by starting with $x_1 = 1$, find x_3 to 5 decimal places.
[Warning: x_3 is not all that close to an actual solution. To get an accurate answer, one would have to go as far as x_6 or x_7. Don't do this.]

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

$$x_{n+1} = x_n - \frac{x_n^4 - 2x - 3}{4x_n^3 - 2}$$

$$x_2 = 1 - \frac{(-4)}{2} = 3$$

$$x_3 = 3 - \frac{81 - 6 - 3}{108 - 2} = 3 - \frac{72}{106}$$

$$= 2.32075$$

NAME: ______________________ STUDENT NO: ______________

2. *[15 marks]* Evaluate the following limits (show all your work).

[4] (a) $\lim\limits_{x\to+\infty} \dfrac{(3-x)(1+2x)}{3x-x^2}$

Method 1: $\frac{\text{poly}}{\text{poly}}$ so same as $\frac{(-x)(2x)}{-x^2} = \boxed{2}$ as $x \to +\infty$

Method 2: (really the same as 1: Divide top and bottom by x^2

$\lim\limits_{x\to\infty} \dfrac{(\frac{3}{x}-1)(\frac{1}{x}+2)}{\frac{3}{x}-1} = \dfrac{(-1)(2)}{-1} = \boxed{2}$

Method 3: $\dfrac{(3-x)(1+2x)}{x(3-x)} = \dfrac{1+2x}{x}$, $x \neq 3$, $\to 2$ as $x \to \infty$.

Method 4: L'Hopital twice.

[5] (b) $\lim\limits_{x\to0^+} \sqrt{x}\ln x = \lim\limits_{x\to0^+} \dfrac{\ln x}{\frac{1}{\sqrt{x}}} = \dfrac{-\infty}{\infty}$

$= \lim\limits_{x\to0^+} \dfrac{\frac{1}{x}}{-\frac{1}{2}x^{-3/2}}$

$= \lim\limits_{x\to0^+} -2x^{\frac{1}{2}} = \boxed{0}$

[6] (c) $\lim\limits_{x\to0^+} \left(\dfrac{1}{e^x-1} - \dfrac{1}{x}\right) = \lim\limits_{x\to0^+} \dfrac{x-e^x+1}{x(e^x-1)} = \dfrac{0}{0}$

$= \lim\limits_{x\to0^+} \dfrac{1-e^x}{e^x-1+xe^x} = \dfrac{0}{0}$

$= \lim\limits_{x\to0^+} \dfrac{-e^x}{e^x+e^x+xe^x} = \boxed{-\frac{1}{2}}$

NAME: ______________________ STUDENT NO: ______________

3. *[15 marks]*

Let $p = 100 \ln\left(\dfrac{1000}{q} + 1\right)$ be the demand function for a certain good.

[7] (a) Find the marginal revenue at $q = 500$.

$R = pq$

$MR = \dfrac{dR}{dq} = p + q\dfrac{dp}{dq}$

$p(500) = 100 \ln(3)$

$\dfrac{dp}{dq} = 100 \dfrac{d}{dq}\left[\ln(1000+q) - \ln q\right]$

$= 100\left[\dfrac{1}{1000+q} - \dfrac{1}{q}\right]$

$\dfrac{dp}{dq} = \dfrac{-100{,}000}{q(q+1000)}$

$\left.\dfrac{dp}{dq}\right|_{q=500} = -\dfrac{100{,}000}{500 \times 1500} = -\dfrac{2}{15}$

$\left.\dfrac{dR}{dq}\right|_{q=500} = 100\ln 3 - \dfrac{500 \times 2}{15} = 100\left(\ln 3 - \dfrac{2}{3}\right) \approx \boxed{43.19}$

[8] (b) Find the point elasticity of demand at $q = 500$ and say whether it is elastic, inelastic or of unit elasticity.

$\eta = \dfrac{p}{q\frac{dp}{dq}} = \dfrac{100\ln 3}{500\left(-\frac{2}{15}\right)} = -\dfrac{3\ln 3}{2}$

$\approx \boxed{-1.65}$

$|\eta| > 1$ so demand is $\boxed{\text{elastic}}$

or $MR = p\left(1 + \dfrac{1}{\eta}\right)$

$43.19 = 100\ln 3\left(1 + \dfrac{1}{\eta}\right)$ solve for η.

NAME: ______________________ STUDENT NO: ______________

4. *[14 marks]*
The equation $e^{2x} + e^y = 11x - 2y + 2$ defines y implicitly as a function of x near the point $(x, y) = (0, 0)$.

[7] (a) Find an expression for y' in terms of x and y near this point, **and** evaluate y' at $(0, 0)$.

$$2e^{2x} + e^y y' = 11 - 2y'$$

$$y'(e^y + 2) = 11 - 2e^{2x}$$

$$\boxed{y' = \frac{11 - 2e^{2x}}{2 + e^y}}$$

At $(0,0)$ $\quad y' = \dfrac{11-2}{2+1} = \boxed{3}$

[7] (b) Find y'' at $(0, 0)$.

Either $\quad 4e^{2x} + e^y (y')^2 + e^y y'' = -2y''$

At $(0,0)$ $\quad 4 + 9 + y'' = -2y''$

$$13 = -3y''$$

$$\boxed{y'' = -13/3}$$

Or

$$y'' = \frac{(2 + e^y)(-4e^{2x}) - (11 - 2e^{2x})e^y y'}{(2 + e^y)^2}$$

At $(0,0)$ $\quad y'' = \dfrac{3 \cdot (-4) - (11-2) \cdot 3}{(2+1)^2}$

$$= -\frac{13}{3} \quad \text{as before}$$