A

Department of Mathematics
University of Toronto
**Tuesday, Jan. 10, 2017, 6:10-8:00 PM**
MAT 133Y TERM TEST #2
Calculus and Linear Algebra for Commerce
Duration: 1 hour 50 minutes

Soln.

**Aids Allowed:** A TI-30X IIS calculator, to be supplied by student. **No other calculator is permitted.**

**Instructions:** Fill in the information on this page, and make sure your test booklet contains 10 pages. In addition, you should have a **multiple-choice answer sheet**, on which you should fill in your name, number, tutorial time, tutorial room, and tutor's name.

This test consists of 10 multiple choice questions, and 4 written-answer questions.
For the **multiple choice questions** you can do your rough work in the test booklet, but you must record your answer by circling the appropriate letter **on the answer sheet** with your pencil. Each correct answer is worth 4 marks; a question left blank, or an incorrect answer, or two answers for the same question is worth 0. For the **written-answer questions**, present your solutions in the space provided. The value of each written-answer question is indicated beside it.

ENCLOSE YOUR FINAL ANSWER IN A BOX AND WRITE IT IN INK.

**TOTAL MARKS: 100**

FAMILY NAME: ______________________

GIVEN NAME: ______________________

STUDENT NO: ______________________

SIGNATURE: ______________________

TUTORIAL TIME and ROOM: ______________________

REGCODE and TIMECODE: ______________________

T.A.'S NAME: ______________________

| Regcode | Timecode | Room | Regcode | Timecode | Room |
|---|---|---|---|---|---|
| T0101 | M9A | HA316 | T0502 | W3B | PB255 |
| T0102 | M9B | HA401 | T0503 | W3C | UC87 |
| T0103 | M9C | HA410 | T0601 | R4A | BA3012 |
| T0104 | M9D | LM157 | T0602 | R4B | AP120 |
| T0201 | M3A | MS4171 | T0603 | R4C | BA B024 |
| T0202 | M3B | UC87 | T0604 | R4D | BF215 |
| T0203 | M3C | GB303 | T0701 | F2A | BA2145 |
| T0204 | M3D | WB119 | T0702 | F2B | BA2155 |
| T0301 | T3A | HA401 | T0703 | F2C | BA2165 |
| T0302 | T3B | UC52 | T0801 | F3A | BA2145 |
| T0303 | T3C | ES B142 | T0802 | F3B | BA2155 |
| T0304 | T3D | GB303 | T0803 | F3C | BA2165 |
| T0401 | W9A | AB114 | T5101 | M5A | AB114 |
| T0402 | W9B | BA2159 | T5102 | M5B | AP120 |
| T0403 | W9C | BA2175 | T5103 | M5C | BA3116 |
| T0501 | W3A | GB248 | T5104 | M5D | HA316 |

| FOR MARKER ONLY | |
|---|---|
| Multiple Choice | |
| B1 | |
| B2 | |
| B3 | |
| B4 | |
| TOTAL | |

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## PART A. Multiple Choice

1. *[4 marks]*

If $f(x) = (x^2 + 5x + 3)^{\frac{1}{2}}$, $f'(1) =$

A. $\frac{7}{6}$

B. $\frac{1}{6}$

C. $\frac{8}{6}$

D. 3

E. 21

$f'(x) = \frac{1}{2}(x^2+5x+3)^{-\frac{1}{2}} \cdot (2x+5)$

$f'(1) = \frac{1}{2} \cdot 9^{-\frac{1}{2}} \cdot 7 = \frac{1}{2} \cdot \frac{1}{3} \cdot 7 = \frac{7}{6}$ A

2. *[4 marks]*

If $y = \dfrac{\sqrt{2x+1}}{2-x}$, then at $x = 1$

$\dfrac{dy}{dx} =$

A. $\sqrt{3}$

B. $\dfrac{7}{\sqrt{3}}$

C. $\dfrac{4}{\sqrt{3}}$

D. $\dfrac{-\sqrt{3}}{4}$

E. $\dfrac{-\sqrt{3}}{7}$

$$\frac{dy}{dx} = \frac{(2-x) \cdot \frac{1}{\cancel{2}\sqrt{2x+1}} \cdot \cancel{2} - \sqrt{2x+1} \cdot (-1)}{(2-x)^2}$$

$$= \frac{(2-x) + (2x+1)}{(2-x)^2 \sqrt{2x+1}}$$

$$= \frac{3+x}{(2-x)^2\sqrt{2x+1}}$$

$$= \frac{4}{1\sqrt{3}} \text{ at } x = 1$$

C

A

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3. *[4 marks]*

If $f(x) = \ln(x^2) - e^5$, $f'(2) =$

$f(x) = 2\ln x - e^5$

$f'(x) = \frac{2}{x}$

$f'(2) = 1$ E

A. $2\ln 2 - e^5$

B. $2\ln 2$

C. $4\ln 2 - 5e^4$

D. 2

E. 1

4. *[4 marks]*

For $y = \ln\left(\ln\left(\frac{1}{x}\right)\right)$, find $y'$ when $x = e^{-1}$.

A. $e$

B. $-e$

C. $\frac{1}{e}$

D. $\frac{-1}{e}$

E. 1

$y' = \frac{1}{\ln\left(\frac{1}{x}\right)} \cdot \frac{1}{\frac{1}{x}} \cdot -\frac{1}{x^2}$

$= \frac{1}{\ln e} \cdot \frac{1}{e} \cdot (-e^2)$

$= -e$ B

Alternatively,

$e^y = \ln\left(\frac{1}{x}\right) = -\ln x$

$e^y y' = -\frac{1}{x}$

When $x = e^{-1}$, $e^y = -\ln x = -\ln(e^{-1}) = 1$

So $y' = -\frac{1}{\frac{1}{e}} = -e$ B again

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5. *[4 marks]*

If $y = \dfrac{(1+x)\sqrt{4+8x+x^4}}{(1+6x)^2}$, then at $x = 0$, $y' =$

Logarithmic Diff is the only sane way to go.

Note that $y(0) = \sqrt{4} = 2$.

A. 14

B. 21

C. $-10$

D. $-20$

E. 28

$\ln y = \ln(1+x) + \frac{1}{2}\ln(4+8x+x^4) - 2\ln(1+6x)$

$\frac{1}{y}y' = \frac{1}{1+x} + \frac{(8+4x^3)}{2(4+8x+x^4)} - \frac{2\cdot 6}{1+6x}$

at $x=0$

$\frac{1}{2}y' = 1 + \frac{8}{8} - 12 = -10$

$y' = -20$ D

6. *[4 marks]*

$\dfrac{d}{dx}\left(3^{2-x^2}\right) =$

$3^{2-x^2}\ln 3\,(-2x)$ D by chain rule

A. $3^{2-x^2}\ln 3$

B. $3^{2-x^2}(2x)$

C. $e^{\ln 3}(-2x)$

D. $3^{2-x^2}(\ln 3)(-2x)$

E. $\dfrac{-3^{2-x^2}}{2x\ln 3}$

or: $y = 3^{2-x^2}$

$\ln y = (2-x^2)\ln 3$

$\frac{1}{y}y' = -2x\ln 3$

$y' = (-2x\ln 3)y$

$= (-2x\ln 3)3^{2-x^2}$ D again

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7. *[4 marks]*

If $u = f(x)$ and $y = g(u)$ and

| | |
|---|---|
| $f(7) = 0$ | $g(0) = 7$ |
| $f(0) = 7$ | $g(7) = 0$ |
| $f'(0) = 3$ | $g'(0) = 1$ |
| $f'(7) = 2$ | $g'(7) = 5$ |

then at $x = 0$, $\dfrac{dy}{dx} =$

A. 2

B. 3

C. 10

D. 14

E. 15

$y(x) = g(u) = g(f(x))$
$y'(x) = g'(f(x))\, f'(x)$
$y'(0) = g'(f(0))\, f'(0)$
$= g'(7)\, 3$
$= 5 \cdot 3 = 15$ E

8. *[4 marks]*

Let $p$ represent price and $q$ represent quantity. Given that $\dfrac{dp}{dq} = q^3 \ln(2q+1)$ and that when $q = 2$, $p = 100$, find the Marginal Revenue at $q = 2$.

A. $8 \ln 5$

B. $12 \ln 5 + \dfrac{16}{5}$

C. 200

D. $16 \ln 5 + 100$

E. $800 \ln 5$

$r = pq$
$\dfrac{dr}{dq} = p + q\dfrac{dp}{dq}$
At $q = 2$, $p = 100$ and $\dfrac{dp}{dq} = 8 \ln 5$
$\dfrac{dr}{dq} = 100 + 16 \ln 5$ D

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9. *[4 marks]*

Let $C$ be consumption at annual income $I$, where $C(I) = 3\sqrt{I} + 700$.
The Marginal Propensity to Save at an annual income of $40,000$ is closest to:

A. 0.9925

B. 0.9998

C. 0.0025

D. 0.0075

E. 0.1300

$$S = I - C$$
$$= I - 3\sqrt{I} - 700$$
$$\frac{dS}{dI} = 1 - \frac{3}{2\sqrt{I}} = 1 - \frac{3}{2\sqrt{40,000}} \quad \text{at } I = 40,000$$
$$\approx 1 - \frac{3}{2 \cdot 200} = 1 - \frac{3}{400}$$
$$\approx .9925 \quad \text{A}$$

10. *[4 marks]*

Using Newton's Method, and starting with $x_1 = 1/2$, find the third approximation, that is $x_3$, of the root of $x^4 = 3x - 1$ that lies between 0 and 1. The third approximation, that is $x_3$, is closest to:

A. 0.3477

B. 0.3376

C. 0.3333

D. 1.6207

E. 0.3250

$$f(x) = x^4 - 3x + 1 = 0$$
$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$
$$= x_n - \frac{x_n^4 - 3x_n + 1}{4x_n^3 - 3}$$
$$x_{n+1} = \frac{3x_n^4 - 1}{4x_n^3 - 3}$$
$$x_2 = \frac{3x_1^4 - 1}{4x_1^3 - 3}$$
$$x_2 = \frac{\frac{3}{16} - 1}{\frac{4}{8} - 3} = \frac{-\frac{13}{16}}{-\frac{20}{8}} = \frac{13}{16} \cdot \frac{8}{20} = \frac{13}{40}$$
$$x_3 = \frac{3\left(\frac{13}{40}\right)^4 - 1}{4\left(\frac{13}{40}\right)^3 - 3} \approx .33763\ldots \quad \text{B}$$

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## PART B. Written-Answer Questions

1. *[15 marks]*

*[7]* (a) Given $f(x) = \begin{cases} \ln x + b & \text{if } x > 1 \\ \dfrac{1}{x^2} & \text{if } x = 1 \\ e^{x+a} & \text{if } x < 1 \end{cases}$

find the values of $a$ and $b$ such that $f$ is continuous at $x = 1$.

For $f$ to be cont at $x=1$

$$\lim_{x\to 1^-} f(x) = f(1) = \lim_{x \to 1^+} f(x)$$

$$(\ln 1) + b = \frac{1}{1} = e^{1+a}$$

So $\boxed{b=1}$ and $1+a=0$ and $\boxed{a=-1}$

*[8]* (b) Solve the following inequality for $x$.

Let $f(x) = \dfrac{(x-5)\ln(x-2)}{|x-4|} \le 0$

Difficulties are $f(x)=0$ at $x=5$ and $x=3$
and $f$ not cont. at $x=4$.
(of course, $f$ is not defined at all for $x \le 2$.)

| Interval | Point x | Sign of f(x) | |
|---|---|---|---|
| (2, 3) | 2.5 | − − + | + |
| (3, 4) | 3.5 | − + + | − |
| (4, 5) | 4.5 | − + + | − |
| (5, ∞) | 1000 | + + + | + |

Since it suffices to check one pt. in each interval,
$f(x) < 0$ on $(3,4) \cup (4,5)$ and $=0$ at $x=3$ and $x=5$
So the $f(x) \le 0$ on $\boxed{[3,4) \cup (4,5]}$

or $\boxed{3 \le x < 4 \text{ or } 4 < x \le 5}$,

or $\boxed{\text{on } [3,5] \text{ with } x=4 \text{ excluded}}$

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2. *[17 marks]*

The demand equation for Mr. Smith's Canadian-Tuxedo Plaid Shirts is given by $q = \sqrt{3000 - p^2}$, where the price $p$ is in U.S. dollars (and $0 < p \leq \sqrt{3000}$).

*[10]* (a) Find the point elasticity of demand when the price of a shirt is \$40.

$$\eta = \frac{\frac{dq}{q}}{\frac{dp}{p}} = \frac{p}{q}\frac{dq}{dp} = \frac{p}{q}\cdot\frac{1}{2\sqrt{3000-p^2}}(-2p) = -\frac{p^2}{q^2}$$

[This can be gotten more easily by

$$q^2 = 3000 - p^2 \Rightarrow q\frac{dq}{dp} = -p \Rightarrow \frac{dq}{dp} = -\frac{p}{q}$$

$$\Rightarrow \eta = \frac{p}{q}\frac{dq}{dp} = -\frac{p^2}{q^2}]$$

When $p = 40$, $q = \sqrt{3000 - 1600} = \sqrt{1400}$

$$\eta = -\frac{p^2}{q^2} = \frac{-1600}{1400} = \boxed{-\frac{8}{7}}$$

*[3]* (b) When the price is \$40, is the demand elastic, inelastic or of unit elasticity?

$|\eta| = \frac{8}{7} > 1$ so demand is $\boxed{\text{elastic}}$

*[4]* (c) At what positive values of $p$ do we have elastic demand?

$|\eta| = \frac{p^2}{q^2} = \frac{p^2}{3000-p^2} > 1$ and since $p \leq \sqrt{3000}$, $3000 - p^2 \geq 0$

so $p^2 > 3000 - p^2$

$2p^2 > 3000$

$p^2 > 1500$

$\boxed{p > \sqrt{1500}}$ gives elastic demand

$\boxed{p > 38.73}$

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3. *[12 marks]*

$x + \ln y = x^2 + y^2 - 3$ defines $y$ as a function of $x$ near the point $(x, y) = (2, 1)$.
Find $y''$ at $x = 2$, $y = 1$. Note that $(2,1)$ satisfies the equation

$(*)$ $1 + \frac{1}{y} y' = 2x + 2y y'$ so at $x = 2, y = 1$

$1 + y' = 4 + 2y'$

$-3 = y'$

Differentiating through $(*)$ again

$\left(-\frac{1}{y^2} y'\right) y' + \frac{1}{y} y'' = 2 + 2y'y' + 2y y''$

$-\frac{(y')^2}{y^2} + \frac{1}{y} y'' = 2 + 2(y')^2 + 2y y''$

$-9 + y'' = 2 + 18 + 2y''$

$\boxed{-29 = y''}$

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Alternatively, starting from $(*)$:

$y'\left(\frac{1}{y} - 2y\right) = 2x - 1$

$y' = \frac{2x-1}{\frac{1}{y} - 2y} = \frac{3}{-1} = -3$ at $(2,1)$.

$y'' = \frac{\left(\frac{1}{y} - 2y\right)2 - (2x-1)\left(-\frac{1}{y^2} y' - 2y'\right)}{\left(\frac{1}{y} - 2y\right)^2}$

$= \frac{-2 - 3(3+6)}{1} = \boxed{-29}$ as before.

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4. *[16 marks]*

In each of the problems below, calculate the limit if it exists or show that it does not. [Note that L'Hôpital's rule is not always the best way to proceed, even where it could be applied.]

*[5]* (a) $\lim_{x\to\infty} \frac{x^2+1}{x\ln x}$ $\quad \frac{\infty}{\infty}$

$$= \lim_{x\to\infty} \frac{2x}{\ln x + \frac{x}{x}} = \lim_{x\to\infty} \frac{2x}{1+\ln x} \quad \frac{\infty}{\infty}$$

$$= \lim_{x\to\infty} \frac{2}{0+\frac{1}{x}} = \lim_{x\to\infty} 2x = \boxed{\infty \text{ (or does not exist)}}$$

*[6]* (b) $\lim_{x\to 2}(x-1)^{\frac{1}{2-x}}$

Let $y = (x-1)^{\frac{1}{2-x}}$ $\quad \ln y = \frac{\ln(x-1)}{2-x}$

$$\lim_{x\to 2} \ln y = \lim_{x\to 2} \frac{\ln(x-1)}{2-x} \quad \frac{0}{0}$$

$$= \lim_{x\to 2} \frac{\frac{1}{x-1}}{-1} = \lim_{x\to 2} -\frac{1}{x-1} = -1$$

$\ln y \to -1$ so $y = e^{\ln y} \to \boxed{e^{-1} = \frac{1}{e}}$

*[5]* (c) $\lim_{x\to\infty} \frac{2^x + 4e^x + 6}{e^x}$

$$= \lim_{x\to\infty} \left(\frac{2^x}{e^x} + \frac{4e^x}{e^x} + \frac{6}{e^x}\right)$$

$$= \lim_{x\to\infty} \left[\left(\frac{2}{e}\right)^x + 4 + \frac{6}{e^x}\right]$$

$\frac{6}{e^x} \to 0$ because $e^x \to \infty$

$\left(\frac{2}{e}\right)^x \to 0$ because $\frac{2}{e} < 1$

So $= \boxed{4}$

Note: L'Hôpital applies but you get $\frac{2^x \ln 2 + 4e^x}{e^x}$, then $\frac{2^x (\ln 2)^2 + 4e^x}{e^x}$

and after $n$ times: $\lim_{x\to\infty} \frac{2^x (\ln 2)^n + 4e^x}{e^x}$ no improvement.