Soln.

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Department of Mathematics
University of Toronto

Tuesday, January 19, 2016, 6:10-8:00 PM
MAT 133Y TERM TEST #2

Calculus and Linear Algebra for Commerce
Duration: 1 hour 50 minutes

Aids Allowed: A non-graphing calculator, with empty memory, to be supplied by student.

Instructions: Fill in the information on this page, and make sure your test booklet contains 10 pages. In addition, you should have a **multiple-choice answer sheet**, on which you should fill in your name, number, tutorial time, tutorial room, and tutor's name.

This test consists of 10 multiple choice questions, and 4 written-answer questions.
For the **multiple choice questions** you can do your rough work in the test booklet, but you must record your answer by circling the appropriate letter **on the answer sheet** with your pencil. Each correct answer is worth 4 marks; a question left blank, or an incorrect answer, or two answers for the same question is worth 0. For the **written-answer questions**, present your solutions in the space provided. The value of each written-answer question is indicated beside it.

ENCLOSE YOUR FINAL ANSWER IN A BOX AND WRITE IT IN INK.

TOTAL MARKS: 100

FAMILY NAME: ______________________

GIVEN NAME: ______________________

STUDENT NO: ______________________

SIGNATURE: ______________________

TUTORIAL TIME and ROOM: ______________________

REGCODE and TIMECODE: ______________________

T.A.'S NAME: ______________________

Regcode	Timecode	Room	**Regcode**	Timecode	Room
T0101A	M9A	BA2135	T0501B	W3B	SS2105
T0101B	M9B	BA2165	T0501C	W3C	UC52
T0101C	M9C	BA1240	T0601A	R4A	BL112
T0101D	M9D	BA2139	T0601B	R4B	BL114
T0201A	M3A	BA B024	T0601C	R4C	SS562
T0201B	M3B	RW142	T0601D	R4D	UC114
T0201C	M3C	WO25	T0701A	F2A	AP120
T0201D	M3D	WW119	T0701B	F2B	BF323
T0301A	T3A	ES4001	T0701C	F2C	LM155
T0301B	T3B	HA316	T0801A	F3A	AP120
T0301C	T3C	SS1086	T0801B	F3B	BF323
T0301D	T3D	SS2111	T0801C	F3C	LM155
T0401A	W9A	BA2195	T5101A	M5A	AP120
T0401B	W9B	AP120	T5101B	M5B	BA2175
T0401C	W9C	LM155	T5101C	M5C	BA2185
T0401D	W9D	BA2159	T5101D	M5D	SS2111
T0501A	W3A	HA316			

FOR MARKER ONLY	
Multiple Choice	
B1	
B2	
B3	
B4	
TOTAL	

NAME: ______________________ STUDENT NO: ______________

PART A. Multiple Choice

1. *[4 marks]*

If $A = \begin{bmatrix} 1 & 1 \\ 4 & 3 \end{bmatrix}$ and $C = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$ then $A^{-1}C =$

A. $\begin{bmatrix} -3 \\ 5 \end{bmatrix}$

B. $\begin{bmatrix} -4 \\ 3 \end{bmatrix}$

C. $\begin{bmatrix} -4 \\ 5 \end{bmatrix}$

D. $\begin{bmatrix} 2 \\ 5 \end{bmatrix}$

E. $\begin{bmatrix} 4 \\ -5 \end{bmatrix}$

$$\left(\begin{array}{cc|cc} 1 & 1 & 1 & 0 \\ 4 & 3 & 0 & 1 \end{array}\right) \xrightarrow{R_2 \to -4R_1 + R_2} \left(\begin{array}{cc|cc} 1 & 1 & 1 & 0 \\ 0 & -1 & -4 & 1 \end{array}\right)$$

$$\xrightarrow[R_1 \to -R_2 + R_1]{R_2 \to -R_2} \left(\begin{array}{cc|cc} 1 & 0 & -3 & 1 \\ 0 & 1 & 4 & -1 \end{array}\right) \qquad A^{-1} = \begin{pmatrix} -3 & 1 \\ 4 & -1 \end{pmatrix}$$

$$A^{-1}C = \begin{pmatrix} -3 & 1 \\ 4 & -1 \end{pmatrix}\begin{pmatrix} 1 \\ -1 \end{pmatrix} = \begin{pmatrix} -3-1 \\ 4+1 \end{pmatrix} = \begin{pmatrix} -4 \\ 5 \end{pmatrix}$$

C

2. *[4 marks]*

Consider

$$f(x) = \begin{cases} -1, & x < 0 \\ 0, & x = 0 \\ 1, & x > 0 \end{cases}$$

Which of the following does not exist?

A. $\lim_{x\to 0^+} f(x)$ $= \lim_{x \to 0^+} 1 = 1$

B. $\lim_{x\to -\infty} f(x)$ $= \lim_{x \to -\infty} -1 = -1$

C. $\lim_{x\to 0^-} f(x)$ $= \lim_{x \to 0^-} -1 = -1$

D. $\lim_{x\to 0} [f(x)]^2$ $\quad [f(x)]^2 = \begin{cases} 1 & x \neq 0 \\ 0 & x = 0 \end{cases} \quad \lim_{x\to 0} [f(x)]^2 = 1$

E. $\lim_{x\to 0} f(x)$ from A+B $\lim_{x \to 0^+} \neq \lim_{x \to 0^-}$ so $\lim_{x\to 0}$ does not exist

E

NAME: ______________________ STUDENT NO: ______________

3. *[4 marks]*

$$\lim_{x\to -1} \frac{2x^2-6x-8}{x^2-1} =$$

$\lim_{x\to -1} \frac{2(x+1)(x-4)}{(x+1)(x-1)} = \lim_{x\to -1} \frac{2(x-4)}{x-1}$

$= \frac{2(-5)}{-2} = 5$ $\boxed{C}$

or L'Hôp $\frac{0}{0}$ so

$= \lim_{x\to -1} \frac{4x-6}{2x} = \frac{-10}{-2} = 5$ as before

C

A. 2

B. 0

C. 5

D. 1

E. does not exist

4. *[4 marks]*

If

$$f(x) = \begin{cases} a-1, & x>2 \\ \frac{-1}{b}, & x=2 \\ 1-x, & x<2 \end{cases} \qquad (a \text{ and } b \text{ constants})$$

is continuous at $x=2$, then which of the following is true?

$\lim_{x\to 2^-} f(x) = f(2) = \lim_{x\to 2^+} f(x)$ (cont.)

$a-1 = -\frac{1}{b} = -1$

so $b=1$ and $a-1=-1$ so $a=0$

A

A. $a=0$ and $b=1$

B. $a=0$ and $b=-1$

C. $a=2$ and $b=1$

D. $a=2$ and $b=-1$

E. there are no real values of a and b that make f continuous at $x=2$

NAME: ____________________ STUDENT NO: ____________________

5. *[4 marks]*

If $f(x) = 5x - \sqrt{x}$, then $f'(4)$ is closest to

A. 4.75

B. 18

C. 4.5

D. 5.25

E. 5.5

$f'(x) = 5 - \frac{1}{2\sqrt{x}}$

$f'(4) = 5 - \frac{1}{4} = 4.75$

A

6. *[4 marks]*

Let $g(x) = e^{5x} \ln\left(3x^4 + 1\right)$. Then $g'(x) =$

A. $\dfrac{5e^{5x}}{12x^3}$

B. $\dfrac{60x^3e^{5x}}{3x^4+1}$

C. $5e^{5x}\ln\left(3x^4+1\right) + \dfrac{12x^3e^{5x}}{3x^4+1}$

D. $5e^{5x}\ln\left(3x^4+1\right) + \dfrac{e^{5x}}{3x^4+1}$

E. $5\ln\left(3x^4+1\right) + \dfrac{60x^3}{3x^4+1}$

$g'(x) = 5e^{5x}\ln(3x^4+1) + \frac{e^{5x} \cdot 12x^3}{3x^4+1}$

C

NAME: ______________________ STUDENT NO: ______________

7. *[4 marks]*

If $y = \sqrt[4]{\dfrac{4s+5}{2s^2+11}}$ then at $s=0$, $\dfrac{dy}{ds}$ is closest to

A. 2.32

B. 0.164

C. 4.07

D. 0.011

E. 0.817

$\ln y = \frac{1}{4}\left[\ln(4s+5) - \ln(2s^2+11)\right]$

$\frac{1}{y} y' = \frac{1}{4}\left[\frac{4}{4s+5} - \frac{4s}{2s^2+11}\right]$

$y'(0) = y(0)\cdot\frac{1}{4}\cdot\frac{4}{5} = \frac{y(0)}{5}$

But $y(0) = \left(\frac{5}{11}\right)^{\frac{1}{4}}$

$y'(0) = \frac{1}{5}\left(\frac{5}{11}\right)^{\frac{1}{4}} \approx .1642$

B

8. *[4 marks]*

Given the consumption function $C(I) = e^{I^2+5I}\left(\dfrac{I}{I^2+10I}\right)$ find
(the marginal propensity to consume) + (the marginal propensity to save).
Hint: Read this question carefully before answering.

A. $e^{I^2+5I}(2I+5)\left(\dfrac{-2I^2+4I}{(I^2+10I)^2}\right)$

B. 1

C. $\dfrac{e^{I^2+5I}}{2I+10} + \dfrac{3-5I}{(I^2+10I)^2}$

D. 0

E. $e^{I^2+5I}\left(\dfrac{I}{I^2+10I}\right) + e^{I^2+5I}\left(\dfrac{2I^2-5I+5}{(I^2+10I)^2}\right)$

$I = C + S$

$1 = \frac{dC}{dI} + \frac{dS}{dI}$

B

NAME: ____________________ STUDENT NO: ____________

9. *[4 marks]*

$\lim_{x\to\infty} \frac{4x+3x^6}{2x^6+x^2} =$

$\lim_{x\to\infty} \frac{\cancel{x^6}\left(\frac{4}{x^5}+3\right)}{\cancel{x^6}\left(2+\frac{1}{x^4}\right)} = \frac{3}{2}$ C

or by L'Hôp. $\frac{\infty}{\infty}$

$= \lim_{x\to\infty} \frac{4+18x^5}{12x^5+2x} = \lim_{x\to\infty} \frac{90x^4}{60x^4+2}$

$= \lim_{x\to\infty} \frac{360\cancel{x^3}}{240\cancel{x^3}} = \frac{360}{240} = \frac{3}{2}$ C

A. 1

B. 2

C. $\frac{3}{2}$

D. 4

E. 3

10. *[4 marks]*

Using Newton's method for finding solutions to the equation $f(x) = 0$ to solve $e^{-x} = \ln(1+x^2)$, and recalling that $x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$, starting with $x_0 = 0$ gives x_2 closest to:

$0 = f(x) = \ln(1+x^2) - e^{-x}$

$x_{n+1} = x_n - \frac{\ln(1+x^2) - e^{-x}}{\frac{2x}{1+x^2} + e^{-x}}$

$x_1 = 0 - \frac{\ln 1 - 1}{0+1} = 1$

$x_2 = 1 - \frac{\ln 2 - \frac{1}{e}}{1 + \frac{1}{e}}$

$\approx .7622$ E

A. 0

B. 1

C. 0.7682

D. 0.7652

E. 0.7622

NAME: ______________________ STUDENT NO: ______________

PART B. Written-Answer Questions

1. *[15 marks]*

Solve the inequalities, explaining your answers

[9] (a) $\dfrac{2^x - 1}{x^2 - x - 2} \le 0$

$f(x) = 0$ at $x = 0$ and discont. at $x^2 - x - 2 = (x-2)(x+1) = 0$; i.e. $x = -1$ and $x = 2$.

	x	$f(x)$
$(-\infty, -1)$	-1000	$-$
$(-1, 0)$	$-\frac{1}{2}$	$+$
$(0, 2)$	1	$-$
$(2, \infty)$	1000	$+$

On each of these intervals one pt is sufficient

$\boxed{x \text{ in } (-\infty, -1) \cup [0, 2)}$

or: $\boxed{x < -1 \text{ or } 0 \le x < 2}$

[6] (b) $1 + \ln(x + 2) > 0$

$f(x) = 1 + \ln(x+2)$ is only defined for $x > -2$ and is cont there.

$f(x) = 0$ when $\ln(x+2) = -1$

$x + 2 = \frac{1}{e}$

$x = \frac{1}{e} - 2$

	x	$f(x)$
$(-2, -2 + \frac{1}{e})$	-1.999	large negative
$(-2 + \frac{1}{e}, \infty)$	0	$1 + \ln 2 > 0$

$\boxed{x > -2 + \frac{1}{e}}$

or $(-2 + \frac{1}{e}, \infty)$

or $\boxed{x > -1.632\ldots}$

NAME: ____________________ STUDENT NO: ____________

2. *[15 marks]*

Given the demand function $p = \frac{q^2}{20} + \frac{150\ln(q+5)}{q}$ and $q = \sqrt{m^2+75}$,

$m = 5 \Rightarrow q = 10.$

[8] (a) find the marginal revenue product when $m = 5$.

$r = pq = \frac{q^3}{20} + 150\ln(q+5)$

$\frac{dr}{dq} = \frac{3q^2}{20} + \frac{150}{q+5} = \frac{300}{20} + \frac{150}{15} = 25$ at $q = 10$

$\frac{dq}{dm} = \frac{1 \cdot 2m}{2\sqrt{m^2+75}} = \frac{5}{\sqrt{100}} = \frac{1}{2}$

$\frac{dr}{dm} = \frac{dr}{dq}\frac{dq}{dm} = \boxed{\frac{25}{2} = 12.5}$ at $q = 10.$

[7] (b) find the point elasticity of demand when $m = 5$. Again $q = 10$

Since we know $\frac{dr}{dq}$, the easiest way is

$\frac{dr}{dq} = p\left(1 + \frac{1}{\eta}\right)$

$p = \frac{100}{20} + \frac{150\ln 15}{10} = 5 + 15\ln 15$

$25 = (5 + 15\ln 15)\left(1 + \frac{1}{\eta}\right)$

$\frac{25}{5+15\ln 15} - 1 = \frac{1}{\eta}$

$\eta = \frac{1}{\frac{25}{5+15\ln 15} - 1} \approx \boxed{-2.21}$

Harder is:

$\frac{dp}{dq} = \frac{q}{10} - \frac{150\ln(q+5)}{q^2} + \frac{150}{q(q+5)}$

$= 1 - 1.5\ln 15 + 1 = 2 - 1.5\ln 15$ at $q = 10$

$\eta = \frac{p}{q\frac{dp}{dq}} = \frac{5 + 15\ln 15}{10(2 - 1.5\ln 15)} \approx \boxed{-2.21}$

A

NAME: ____________________ STUDENT NO: ____________________

3. *[16 marks]*

Given that $x^2y^4 + 2x = 8y^3$ defines y implicitly as a function of x near the point $x = 2$ and $y = 1$,

[8] (a) find $\frac{dy}{dx}$ at $x = 2$, $y = 1$.

$$2xy^4 + 4x^2y^3y' + 2 = 24y^2y'$$

$$4 + 16y' + 2 = 24y' \quad \text{at } x=2, y=1$$

$$6 = 8y'$$

$$y' = \frac{6}{8} = \boxed{\frac{3}{4}}$$

or $y'[4x^2y^3 - 24y^2] = -2 - 2xy^4$

$$y' = -\frac{(1+xy^4)}{2x^2y^3 - 12y^2} = -\frac{3}{-4} = \boxed{\frac{3}{4}}$$

[8] (b) find $\frac{d^2y}{dx^2}$ at $x = 2$, $y = 1$.

Starting with the first line in (a): $xy^4 + 2x^2y^3y' + 1 = 12y^2y'$ and differentiating

$$y^4 + 4xy^3y' + 4xy^3y' + 6x^2y^2(y')^2 + 2x^2y^3y'' = 24y(y')^2 + 12y^2y''$$

$$1 + 8\cdot\frac{3}{4} + 8\cdot\frac{3}{4} + 24\cdot\frac{9}{16} + 8y'' = 24\cdot\frac{9}{16} + 12y''$$

$$13 = 4y''$$

$$\boxed{\frac{13}{4} = y''} = 3.25$$

Or: Starting with the last line of (a) and using the quotient rule:

$$y'' = -\left[\frac{(2x^2y^3 - 12y^2)(y^4 + 4xy^3y') - (1+xy^4)(4xy^3 + 6x^2y^2y' - 24yy')}{(2x^2y^3 - 12y^2)^2}\right]$$

$$= -\left[\frac{(8-12)(1 + 8\cdot\frac{3}{4}) - (1+2)(8 + 24\cdot\frac{3}{4} - 24\cdot\frac{3}{4})}{(8-12)^2}\right]$$

$$= -\frac{[(-4)(7) - (3)(8)]}{16} = -\frac{(-52)}{16} = \frac{13}{4} = 3.25$$

as before

NAME: ____________________ STUDENT NO: ____________

4. *[14 marks]*

Find the following limits, showing all your work. (Numerical tabulations don't count.)

[7] (a) $\lim_{x\to 0^+} x(\ln x)^2$ $0 \cdot (\infty)$

$= \lim_{x\to 0^+} \frac{(\ln x)^2}{\frac{1}{x}} = \frac{\infty}{\infty}$

$= \lim_{x\to 0^+} \frac{\frac{2\ln x}{x}}{-\frac{1}{x^2}}$ L'Hôp

$= \lim_{x\to 0^+} -2x\ln x = \lim_{x\to 0^+} -2\frac{\ln x}{\frac{1}{x}}$ LHôp again

$= \lim_{x\to 0^+} \frac{-2\frac{1}{x}}{-\frac{1}{x^2}} = \lim_{x\to 0^+} 2x = \boxed{0}$

[7] (b) $\lim_{x\to\infty} \frac{a^{\frac{1}{x}}-1}{a^{\frac{5}{x}}-1}$, where $a>1$ is constant.

Since $\frac{1}{x}\to 0$ $a^{\frac{1}{x}}$ and $a^{\frac{5}{x}} \to a^0 = 1$; So $\frac{0}{0}$.

L'Hôp : $\lim_{x\to\infty} = \lim_{x\to\infty} \frac{a^{\frac{1}{x}}\ln a\left(-\frac{1}{x^2}\right)}{a^{\frac{5}{x}}(\ln a)\left(-\frac{5}{x^2}\right)}$

$= \lim_{x\to\infty} \frac{1}{a^{4/x}\cdot 5}$; but $a^{\frac{4}{x}} \to a^0$

so $= \boxed{\frac{1}{5}}$