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Department of Mathematics
University of Toronto

Tuesday, January 13, 2015, 6:10-8:00 PM
MAT 133Y TERM TEST #2

Calculus and Linear Algebra for Commerce
Duration: 1 hour 50 minutes

Aids Allowed: A non-graphing calculator, with empty memory, to be supplied by student.

Instructions: Fill in the information on this page, and make sure your test booklet contains 10 pages. In addition, you should have a **multiple-choice answer sheet**, on which you should fill in your name, number, tutorial time, tutorial room, and tutor's name.

This test consists of 10 multiple choice questions, and 4 written-answer questions.
For the **multiple choice questions** you can do your rough work in the test booklet, but you must record your answer by circling the appropriate letter **on the answer sheet** with your pencil. Each correct answer is worth 4 marks; a question left blank, or an incorrect answer, or two answers for the same question is worth 0. For the **written-answer questions**, present your solutions in the space provided. The value of each written-answer question is indicated beside it.

ENCLOSE YOUR FINAL ANSWER IN A BOX AND WRITE IT IN INK.

TOTAL MARKS: 100

FAMILY NAME: ______________________

GIVEN NAME: ______________________

STUDENT NO: ______________________

SIGNATURE: ______________________

TUTORIAL TIME and ROOM: ______________________

REGCODE and TIMECODE: ______________________

T.A.'S NAME: ______________________

Regcode	Timecode	Room	Regcode	Timecode	Room
T0101A	M9A	SS1086	T0501B	W3B	UC261
T0101B	M9B	SS1088	T0601A	R4A	BA2155
T0101C	M9C	SS2128	T0601B	R4B	UC261
T0201A	M3A	UC261	T0601C	R4C	UC330
T0201B	M3B	BL112	T0601D	R4D	BL114
T0201C	M3C	BL114	T0701A	F2A	SS1070
T0201D	M3D	BF215	T0701B	F2B	SS1074
T0301A	T3A	BA2139	T0701C	F2C	SS1083
T0301B	T3B	BA2159	T0701D	F2D	SS2106
T0301C	T3C	WW126	T0801A	F3A	SS1070
T0301D	T3D	AB114	T0801B	F3B	SS1074
T0401A	W9A	LM155	T5101A	M5A	BA1210
T0401B	W9B	LM157	T5101B	M5B	BL114
T0401C	W9C	MP118	T5201A	M6A	SS1088
T0501A	W3A	SS1087			

FOR MARKER ONLY	
Multiple Choice	
B1	
B2	
B3	
B4	
TOTAL	

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PART A. Multiple Choice

1. *[4 marks]*

$$\lim_{x\to\infty} \frac{5+4x-3x^2+7x^3-2x^4+3x^5}{2x^5+7x^4-3x^3+2x^2+x+1} =$$

A. undefined

B. $\frac{5}{2}$

C. 5

D. 3

E. $\frac{3}{2}$

$$\lim_{x\to\infty} \frac{\cancel{x^5}\left(\frac{5}{x^5}+\frac{4}{x^4}-\frac{3}{x^3}+\frac{7}{x^2}-\frac{2}{x}+3\right)}{\cancel{x^5}\left(2+\frac{7}{x}-\frac{3}{x^2}+\frac{2}{x^3}+\frac{1}{x^4}+\frac{1}{x^5}\right)}$$

$$= \frac{3}{2} \quad E$$

2. *[4 marks]*

$$f(x) = \begin{cases} a + 3^{\frac{1}{x-1}} & \text{if } x < 1 \\ 2 & \text{if } x = 1 \\ b^{\frac{1}{x+1}} & \text{if } x > 1 \end{cases}$$

and if f is continuous at all values of x, then:

A. $a =\ \ 2$ and $b = 2$

B. $a =\ \ 1$ and $b = 4$

C. $a = -1$ and $b = 2$

D. $a =\ \ 2$ and $b = 4$

E. $a = -1$ and $b = 4$

$$\lim_{x\to 1^-} f(x) = a + \lim_{x\to 1^-} 3^{\frac{1}{x-1}} \quad \text{since } x-1 \to 0^-,\ \frac{1}{x-1} \to -\infty$$

$$= a + 0 = a$$

$$f(1) = 2$$

$$\lim_{x\to 1^+} f(x) = b^{\frac{1}{2}}$$

So $a = 2 = b^{\frac{1}{2}}$ and $b = 4$

D

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3. *[4 marks]*

If $y = (5x^3 - 1)\ln x$, then at $x = 1$ $\frac{dy}{dx} =$ $15x^2 \ln x + \frac{5x^3 - 1}{x}$

at $x=1$, $= 15 \cdot 1 \cdot 0 + \frac{5 \cdot 1 - 1}{1} = 4$ C

A. 0

B. 15

C. 4

D. 6

E. 9

4. *[4 marks]*

If $f(x) = \frac{x^2 + x}{\sqrt{x+1}}$, $f'(3) =$

$$f'(x) = \frac{\sqrt{x+1}\,(2x+1) - \frac{(x^2+x)}{2\sqrt{x+1}}}{x+1}$$

$$= \frac{\sqrt{4} \cdot 7 - \frac{12}{2 \cdot \sqrt{4}}}{4}$$

$$= \frac{2 \cdot 7 - 3}{4} = \frac{11}{4}$$ D

A. $\frac{17}{2}$

B. $\frac{17}{4}$

C. $\frac{13}{4}$

D. $\frac{11}{4}$

E. $\frac{11}{2}$

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5. *[4 marks]*

Let $f(x) = \sqrt{1+2(g(x))^2}$, $g(1) = 2$, and $g'(1) = 6$. Then $f'(1) =$

A. 3

B. 18

C. 8

D. $\frac{4}{3}$

E. $\frac{3}{2}$

$$f'(x) = \frac{1}{2\sqrt{1+2[g(x)]^2}} \cdot 2 \cdot 2g(x)g'(x)$$

$$= \frac{2g(x)g'(x)}{\sqrt{1+2[g(x)]^2}}, \text{ when } x=1,\ g(1)=2,\ g'(1)=6$$

$$f'(1) = \frac{2 \cdot 2 \cdot 6}{\sqrt{1+2\cdot 2^2}} = \frac{24}{3} = 8 \quad C$$

6. *[4 marks]*

Let $f(x) = 2^{3x^4-1}$. The slope of the tangent line to the graph of $y = f(x)$ at $(1,4)$ is

A. 48

B. 2^{12}

C. $48\ln 2$

D. $4\ln 2$

E. 6

The slope of the tangent line is $f'(1)$.

$$\ln f = (3x^4-1)\ln 2$$

$$\frac{1}{f}f' = 12x^3\ln 2. \quad \text{At } x=1\ f(x)=4$$

So $\frac{1}{4}f'(1) = 12\ln 2$

$$f'(1) = 48\ln 2 \quad C$$

Alternatively: $f'(x) = 2^{3x^4-1}\ln 2 \cdot 12x^3$

$$f'(1) = 2^2 \ln 2 \cdot 12$$

$$= 48\ln 2 \text{ as before}$$

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7. *[4 marks]*

Let $f(x) = (x^2+5)^{\sqrt{x}}$ and $x > 0$. Then $f'(x) =$

A. $\left(\dfrac{\ln(x^2+5)}{2\sqrt{x}} + \dfrac{\sqrt{x}(2x)}{x^2+5}\right)(x^2+5)^{\sqrt{x}}$

B. $\sqrt{x}(x^2+5)^{\sqrt{x}-1}$

C. $(x^2+5)^{\sqrt{x}}(2x)\left(\dfrac{1}{2\sqrt{x}}\right)$

D. $\left(\dfrac{1}{2\sqrt{x}}\right)\left(\dfrac{2x}{x^2+5}\right)$

E. $\dfrac{\ln(x^2+5)}{2\sqrt{x}} + \dfrac{\sqrt{x}(2x)}{x^2+5}$

$\ln f = \sqrt{x}\ \ln(x^2+5)$

$\frac{1}{f} f' = \frac{1}{2\sqrt{x}} \ln(x^2+5) + \frac{\sqrt{x}(2x)}{x^2+5}$

$f' = (x^2+5)^{\sqrt{x}}\left(\frac{1}{2\sqrt{x}} \ln(x^2+5) + \frac{\sqrt{x}(2x)}{x^2+5}\right)$

A

8. *[4 marks]*

If $C(I) = \sqrt{I}$ is the total spent on consumption depending on national income (I), then when $I = 25$, marginal propensity to **save** is

A. 0.9

B. 0.1

C. 0.8

D. 0.2

E. 0.5

$S = I - C$

$\frac{dS}{dI} = 1 - \frac{dC}{dI}$

$= 1 - \frac{1}{2\sqrt{I}}$

$= 1 - \frac{1}{2\sqrt{25}}$ at $I = 25$

$= 1 - \frac{1}{10} = .9$ A

A

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9. *[4 marks]*

$\lim_{x\to 1} \frac{x^{\frac{1}{3}} - 1}{x^2 - 1} =$

0/0

$$\lim_{x\to 1} \frac{\frac{1}{3}x^{-\frac{2}{3}}}{2x} = \frac{1}{6}$$ D

A. undefined

B. $\frac{1}{2}$

C. $\frac{1}{3}$

D. $\frac{1}{6}$

E. 1

10. *[4 marks]*

If $x_1 = 1$ is used as a first estimate for a root of $f(x) = x^5 + x - 1 = 0$, then Newton's method yields the third estimate (to 5 decimal places) $x_3 =$

A. 0.77262

B. 0.76438

C. 0.78597

D. 0.83333

E. 0.80591

$$X_{n+1} = X_n - \frac{f(x_n)}{f'(x_n)}$$

$$= X_n - \frac{X_n^5 + X_n - 1}{5X_n^4 + 1}$$

$$= \frac{4X_n^5 + 1}{5X_n^4 + 1}$$

$$X_2 = \frac{5}{6}$$

$$X_3 = \frac{4\cdot\left(\frac{5}{6}\right)^5 + 1}{5\left(\frac{5}{6}\right)^4 + 1} \approx .76438\,2115$$ B

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PART B. Written-Answer Questions

1. *[10 marks]*

Solve the following inequality for x.

$$\frac{(x-3)\ln x}{e^x - 5} \leq 0$$

The function is only defined for $x > 0$ (because of $\ln x$) and for $e^x - 5 \neq 0$, i.e. $x \neq \ln 5 \approx 1.61$.

The function is 0 only at $x=3$ and $x=1$.

So, on the following intervals, the function is continuous and $\neq 0$:

Interval	point	sign of function
$(0, 1)$	$\frac{1}{2}$	$-$
$(1, \ln 5)$	$\frac{3}{2}$	$+$
$(\ln 5, 3)$	2	$-$
$(3, \infty)$	10^{10}	$+$

since at $x = \frac{1}{2}$: $\dfrac{-\frac{5}{2}\ln\left(\frac{1}{2}\right)}{e^{\frac{1}{2}} - 5} = \dfrac{\frac{5}{2}\ln 2}{\sqrt{e} - 5} = \dfrac{+}{-} = -$ since $\ln 2 > 0$ and $\sqrt{e} < 5$

at $x = \frac{3}{2}$: $\dfrac{-\frac{3}{2}\ln\frac{3}{2}}{e^{3/2} - 5} = \dfrac{-}{-} = +$ since $\ln\frac{3}{2} > 0$ and $e^3 \approx 20$

at $x = 2$: $\dfrac{-\ln 2}{e^2 - 5} < 0$

and at $x =$ very large, everything is > 0.

Answer: $\boxed{(0, 1] \cup (\ln 5, 3]}$

Alternatively $\boxed{0 < x \leq 1 \text{ or } \ln 5 < x \leq 3}$

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2. *[18 marks]*
Evaluate the follwing limits or show they don't exist. (Show your work.)

[6] (a) $\lim_{x\to 1^-} \dfrac{|x-1|}{x^2-3x+2}$

$x \to 1^- \Rightarrow x<1 \Rightarrow |x-1| = 1-x$

We have $\lim_{x\to1^-} \dfrac{1-x}{x^2-3x+2} = \lim_{x\to1^-} \dfrac{1-x}{(x-1)(x-2)} = \lim_{x\to1^-} -\dfrac{1}{x-2} = \boxed{1}$

or: $\dfrac{0}{0}$ $= \lim_{x\to1^-} -\dfrac{1}{2x-3} = \boxed{1}$

[6] (b) $\lim_{x\to\infty} (4x+e^x)^{\frac{2}{x}}$ Let $y = (4x+e^x)^{\frac{2}{x}}$

$\ln y = \dfrac{2\ln(4x+e^x)}{x}$ $\quad \dfrac{\infty}{\infty}$

$\lim_{x\to\infty} \ln y = \lim_{x\to\infty} \dfrac{2(4+e^x)}{4x+e^x} = \lim_{x\to\infty} \dfrac{2e^x}{4+e^x} = \lim_{x\to\infty} \dfrac{2e^x}{e^x} = 2$

or $= \lim_{x\to\infty} \dfrac{2}{4e^{-x}+1} = \dfrac{2}{0+1} = 2$

$\ln y \to 2$, so $y \to \boxed{e^2}$

[6] (b) $\lim_{x\to2} \left[\dfrac{1}{\ln(x-1)} - \dfrac{1}{x-2}\right] = \lim_{x\to2} \dfrac{(x-2)-\ln(x-1)}{(x-2)\ln(x-1)}$ $\quad \dfrac{0}{0}$

$= \lim_{x\to2} \dfrac{1-\frac{1}{x-1}}{\ln(x-1)+\frac{x-2}{x-1}} = \lim_{x\to2} \dfrac{x-2}{(x-1)\ln(x-1)+(x-2)}$ (by algebra only)

L'Hop: $= \lim_{x\to2} \dfrac{1}{\ln(x-1)+\frac{x-1}{x-1}+1} = \boxed{\dfrac{1}{2}}$

There are many other combinations of algebra & L'Hop. that get here correctly.

A

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3. *[17 marks]*

Let $p =$ unit price, $q =$ quantity, and $m =$ number of employees. The demand function is

$$p = 10 + \frac{100}{q} \qquad \text{and} \qquad q = \frac{60m}{\sqrt{11+m^2}}$$

When $m = 5$, $q = 300/6 = 50$ and $p = 12$.

[4] (a) find the marginal revenue.

$$r = pq = 10q + 100$$

$$\frac{dr}{dq} = \boxed{10}$$

[6] (b) find the point elasticity of demand. Is demand elastic, inelastic, or of unit elasiticity?

$$\mu = \frac{\frac{dq}{q}}{\frac{dp}{p}} = \frac{p}{q\frac{dp}{dq}} = \frac{p}{q\left(-\frac{100}{q^2}\right)} = -\frac{pq}{100} = -\frac{600}{100} = \boxed{-6}$$

$|\mu| = 6 > 1$ so demand $\boxed{\text{is elastic}}$

[7] (c) find the marginal revenue product.

$$\frac{dr}{dm} = \frac{dr}{dq}\frac{dq}{dm} \qquad \frac{dr}{dq} = 10$$

$$\frac{dq}{dm} = 60\left[\frac{\sqrt{11+m^2} - \frac{2m\cdot m}{2\sqrt{11+m^2}}}{11+m^2}\right] = \frac{60}{36}\left[6 - \frac{25}{6}\right] = \frac{60\cdot 11}{6\cdot 36} = \frac{55}{18}$$

$$\approx 3.0555\ldots$$

$$\frac{dr}{dm} \approx \boxed{30.555} \text{ or } \boxed{30\tfrac{5}{9}}$$

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4. *[15 marks]*

[7] (a) If $3x + 4y - x^2y^3 = 4$ defines y implicitly as a function of x, find y' as a function of x and y. Then find the equation to the tangent line defined by this curve at $x = 3$, $y = 1$.

$$3 + 4y' - 2xy^3 - 3x^2y^2y' = 0$$

$$\boxed{y' = \frac{2xy^3 - 3}{4 - 3x^2y^2}} = \frac{6-3}{4-27} = -\frac{3}{23} \text{ at } (3,1).$$

$$\boxed{y - 1 = -\frac{3}{23}(x-3)}$$ is the equation of the tangent line at (3,1)

[8] (b) $y \ln x + xe^y = 1$ defines y implicitly as a function of x near the point $x = 1$, $y = 0$. Find y'' at $x = 1$, $y = 0$.

$$y' \ln x + \frac{y}{x} + e^y + xe^y y' = 0 \quad \therefore \text{At } (1,0) \quad e^0 + y' = 0$$

$$\text{so } y' = -1$$

$$\left(y'' \ln x + \frac{y'}{x}\right) + \left(\frac{y'}{x} - \frac{y}{x^2}\right) + e^y y' + \left(e^y y' + x e^y (y')^2 + x e^y y''\right) = 0$$

Substituting $x = 1, y = 0, y' = -1$

$$\cancel{y'' \cdot 0} + \frac{(-1)}{1} + \frac{(-1)}{1} - \cancel{0} + (-1) + (-1) + 1 \cdot (-1)^2 + y'' = 0$$

$$-4 + 1 + y'' = 0$$

$$\boxed{y'' = 3}$$