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Department of Mathematics
University of Toronto

Tuesday, January 14, 2014, 6:10-8:00 PM
MAT 133Y TERM TEST #2

Calculus and Linear Algebra for Commerce
Duration: 1 hour 50 minutes

Aids Allowed: A non-graphing calculator, with empty memory, to be supplied by student.

Instructions: Fill in the information on this page, and make sure your test booklet contains 10 pages. In addition, you should have a **multiple-choice answer sheet**, on which you should fill in your name, number, tutorial time, tutorial room, and tutor's name.

This test consists of 10 multiple choice questions, and 4 written-answer questions.
For the **multiple choice questions** you can do your rough work in the test booklet, but you must record your answer by circling the appropriate letter **on the answer sheet** with your pencil. Each correct answer is worth 4 marks; a question left blank, or an incorrect answer, or two answers for the same question is worth 0. For the **written-answer questions**, present your solutions in the space provided. The value of each written-answer question is indicated beside it.

ENCLOSE YOUR FINAL ANSWER IN A BOX AND WRITE IT IN INK.

TOTAL MARKS: 100

FAMILY NAME: ______________________

GIVEN NAME: ______________________

STUDENT NO: ______________________

SIGNATURE: ______________________

TUTORIAL TIME and ROOM: ______________________

REGCODE and TIMECODE: ______________________

T.A.'S NAME: ______________________

Regcode	Timecode	Room	**Regcode**	Timecode	Room
T0101A	M9A	SS1084	T0501B	W3B	UCA101
T0101B	M9B	SS1086	T0601A	R4A	BA1210
T0101C	M9C	SS1080	T0601B	R4B	BA1220
T0201A	M3A	MP118	T0601C	R4C	AP120
T0201B	M3B	UC52	T0601D	R4D	GB120
T0201C	M3C	UC152	T0701A	F2A	LM157
T0201D	M3D	GB221	T0701B	F2B	BF215
T0301A	T3A	RW143	T0701C	F2C	MP137
T0301B	T3B	BA1210	T0701D	F2D	SS2127
T0301C	T3C	AP120	T0801A	F3A	BF215
T0301D	T3D	WW126	T0801B	F3B	MP118
T0401A	W9A	SS1084	T5101A	M5A	SS2105
T0401B	W9B	SS1088	T5101B	M5B	UC87
T0401C	W9C	LM155	T5201A	M6A	LM162
T0501A	W3A	UC85			

FOR MARKER ONLY	
Multiple Choice	
B1	
B2	
B3	
B4	
TOTAL	

NAME: ______________________ STUDENT NO: ______________

PART A. Multiple Choice

1. *[4 marks]*

If f is defined for all real x except $x = 2$ by

$$f(x) = \begin{cases} 1 & \text{if } x \le 0 \\ x & \text{if } 0 < x \le 1 \\ \frac{1}{2-x} & \text{if } x > 1 \end{cases}$$

then f is not continuous at

The only candidates are $x=0$, $x=1$ and $x=2$.

$\lim_{x\to 0^-} f(x) = 1 \neq \lim_{x\to 0^+} f(x) = \lim_{x\to 0^+} x = 0$

$\boxed{f \text{ discont at } x=0}$

$\lim_{x\to 1^-} f(x) = \lim_{x\to 1^-} x = 1$

$\lim_{x\to 1^+} f(x) = \lim_{x\to 1^+} \frac{1}{2-x} = 1$

$f(1) = 1$ all equal

$\boxed{f \text{ cont at } x=1}$

f undefined at $x=2$

$\boxed{f \text{ discont at } x=2}$

C

A. $x = 1$ and $x = 2$ only

B. $x = 0$ only

C. $x = 0$ and $x = 2$ only

D. $x = 1$ only

E. $x = 0$, $x = 1$, and $x = 2$ only

2. *[4 marks]*

If $y = \dfrac{x}{\ln x}$, then when $x = e^2$, $\dfrac{dy}{dx} =$

$$\frac{\ln x - x\cdot\frac{1}{x}}{(\ln x)^2} = \frac{\ln x - 1}{(\ln x)^2}$$

$$= \frac{\ln(e^2) - 1}{[\ln(e^2)]^2} = \frac{2-1}{2^2} = \frac{1}{4}$$

B

A. $\dfrac{e}{2}$

B. $\dfrac{1}{4}$

C. $\dfrac{e^2}{2}$

D. 0

E. $\dfrac{1}{2}$

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3. *[4 marks]*

If $f(x) = x^2 e^{3x}$, then $f''(1) =$

A. $9e^3$

B. $4e^3$

C. $36e^3$

D. 2

E. $23e^3$

$f'(x) = 2xe^{3x} + 3x^2 e^{3x}$

$f''(x) = 2e^{3x} + 6xe^{3x} + 6xe^{3x} + 9x^2 e^{3x}$

$= e^{3x}(2 + 12x + 9x^2)$

$f''(1) = 23e^3$ E

4. *[4 marks]*

Let $h(x) = f(x) \cdot g(x)$, where

$f(1) = 3 \quad f(5) = 4 \quad g(1) = 5$

$f'(1) = 2 \quad f'(5) = 9 \quad g'(1) = 7$

Then $h'(1) =$

A. 31

B. 63

C. 29

D. 73

E. 30

$h'(x) = f'(x)g(x) + f(x)g'(x)$

$h'(1) = f'(1)g(1) + f(1)g'(1)$

$= 2 \cdot 5 + 3 \cdot 7$

$= 31$ A

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5. *[4 marks]*

Let $h(x) = \sqrt{f(x)}$, where
$f(0) = 25 \quad f(1) = 0$
$f'(0) = 4 \quad f'(1) = 9$
Then $h'(0) =$

$h(x) = [f(x)]^{\frac{1}{2}}$

$h'(x) = \frac{1}{2}[f(x)]^{-\frac{1}{2}} f'(x)$

$h'(0) = \frac{1}{2}[f(0)]^{-\frac{1}{2}} f'(0)$

$= \frac{1}{2} \cdot (25)^{-\frac{1}{2}} \cdot 4$

$= \frac{1}{2 \cdot 5} \cdot 4 = \frac{2}{5}$ D

A. 2

B. 0

C. $\frac{2}{3}$

D. $\frac{2}{5}$

E. $\frac{9}{4}$

6. *[4 marks]*

The Consumption Function is given by: $C = 5 + I - 4\sqrt{I}$. If I is income and S is savings, which of the following is **false**?

A. The marginal propensity to consume $= \dfrac{\sqrt{I} - 2}{\sqrt{I}}$ $MPC = \frac{dC}{dI} = 1 - \frac{2}{\sqrt{I}} = \frac{\sqrt{I}-2}{\sqrt{I}}$ true

B. $S + C = I$ true, by definition

C. The rate of change of savings with respect to I is $\dfrac{2}{\sqrt{I}}$ $\frac{dS}{dI} = 1 - \frac{dC}{dI} = \frac{2}{\sqrt{I}}$ true

D. $\dfrac{dS}{dI} = \dfrac{dC}{dI}$ when $I = 16$ When $I = 16$ $1 - \frac{2}{\sqrt{I}} = 1 - \frac{1}{2} = \frac{1}{2}$ and $\frac{2}{\sqrt{I}} = \frac{2}{\sqrt{16}} = \frac{1}{2}$ so true

E. $S = 5 - 4\sqrt{I}$ The only chance:

$S = I - C = I - (5 + I - 4\sqrt{I})$

$= 4\sqrt{I} - 5$ not $5 - 4\sqrt{I}$.

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7. *[4 marks]*

If Newton's method is used to approximate a zero of $f(x) = x^5 + x + 1$ and the initial approximation is $x_1 = 0$, then $x_3 =$

A. -2

B. 5

C. -1

D. $-\dfrac{5}{6}$

E. $-\dfrac{1}{2}$

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

$$= x_n - \frac{x_n^5 + x_n + 1}{5x_n^4 + 1}$$

$$x_{n+1} = \frac{4x_n^5 - 1}{5x_n^4 + 1}$$

If $x_1 = 0$, $x_2 = -1$

$$x_3 = \frac{4(-1)^5 - 1}{5(-1)^4 + 1} = -\frac{5}{6} \quad \text{D}$$

8. *[4 marks]*

$$\lim_{x\to 1} \frac{(x-1)^2}{e^{x-1} - x} =$$

A. e^{-1}

B. ∞

C. 2

D. 0

E. 1

$$\lim_{x\to 1} \frac{2(x-1)}{e^{x-1} - 1} \quad \text{(because } \tfrac{0}{0}\text{)}$$

$$= \lim_{x\to 1} \frac{2}{e^{x-1}} \quad \text{(again } \tfrac{0}{0}\text{)}$$

$= 2$ by substitution

C

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9. *[4 marks]*

$\lim\limits_{x\to 0} \frac{e^x}{x} - \frac{1}{xe^x} =$

$\infty - \infty$

$\lim\limits_{x\to 0} \frac{e^{2x}-1}{xe^x}$ $\quad \frac{0}{0}$

$= \lim\limits_{x\to 0} \frac{2e^{2x}}{e^x + xe^x} = 2$ D

A. ∞

B. 0

C. e

D. 2

E. 1

or: $\frac{e^x}{x} - \frac{1}{xe^x} = \frac{e^x - e^{-x}}{x}$ $\quad \frac{0}{0}$

$\lim\limits_{x\to 0} \frac{e^x - e^{-x}}{x} = \lim\limits_{x\to 0} \frac{e^x + e^{-x}}{1} = 2$ as before

10. *[4 marks]*

The asymptotes of $f(x) = xe^{-x}$ are:

A. $x = 0$ and $y = 0$

B. $x = 0$ and $y = 1$

C. $y = 0$ only

D. $y = 1$ only

E. f has no asymptotes

f is cont everywhere so has no V.A.

$\lim\limits_{x\to -\infty} f(x) = -\infty$

but $\lim\limits_{x\to \infty} f(x) = \lim\limits_{x\to \infty} \frac{x}{e^x}$

$= \lim\limits_{x\to \infty} \frac{1}{e^x} = 0$

so $y=0$ is H.A. C

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PART B. Written-Answer Questions

1. *[11 marks]*

Solve the inequality $\dfrac{x-3}{\ln x - 2} \geq 0$.

$\dfrac{x-3}{\ln x - 2}$ is only defined for $x > 0$.

It is cont on $x > 0$ as long as $\ln x \neq 2$, that is $x \neq e^2$.

Hence $\dfrac{x-3}{\ln x-2}$ is cont. and unequal to zero, for $x > 0$, except at $x = 3$ and $x = e^2 > 3$.

	f
$(0,3)$	+
$(3,e^2)$	−
(e^2,∞)	+

Inside each of these intervals f is cont. and never zero, hence cannot change sign. A single test point will do.

In $(0,3)$ try 1:

$$\frac{1-3}{\ln 1 - 2} = \frac{-2}{-2} = 1 > 0$$

In $(3,e^2)$ try 4: either by calculator or by $4 < e^2$ the denom < 0 and $x - 3 > 0$.

In (e^2,∞), numerator and denom obviously > 0.

The solution is

$$(0,3] \cup (e^2,\infty)$$

or: $\{0 < x \leq 3 \text{ or } e^2 < x\}$

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2. *[16 marks]*

If the revenue function is given by: $r = \dfrac{3000q}{q+100}$ and $q = 100m - m^2$ where m is the number of employees needed to produce q units, find

[4] (a) the marginal revenue when $q = 900$.

$$MR = \frac{dr}{dq} = 3000\left[\frac{(q+100)-q}{(q+100)^2}\right] = \frac{3\times10^5}{(q+100)^2}$$

$$\frac{dr}{dq} = \frac{3\times10^5}{1000^2} = \boxed{\frac{3}{10} = .3} \text{ at } q = 900$$

[4] (b) the marginal revenue product when $m = 10$. When $m=10$, $q = 1000 - 100 = 900$

$$\frac{dr}{dm} = \frac{dr}{dq}\frac{dq}{dm}$$

$$= .3\,\frac{dq}{dm} \text{ when } m=10. \quad \frac{dq}{dm} = 100 - 2m = 80 \text{ when } m=10$$

$$= .3\cdot 80 = \boxed{24}$$

[4] (c) elasticity of demand when $q = 900$.

$$pq = r = \frac{3000q}{q+100} \text{ so } p = \frac{3000}{q+100} \text{ the demand fcn.}$$

$$\eta = \frac{\frac{dq}{q}}{\frac{dp}{p}} = \frac{p}{q\frac{dp}{dq}} = \frac{p}{q\left(-\frac{3000}{(q+100)^2}\right)} = -\frac{\frac{3000}{q+100}}{\frac{3000q}{(q+100)^2}} = -\left(\frac{q+100}{q}\right)$$

$$= -\frac{1000}{900} = \boxed{-\frac{10}{9}}$$

[4] (d) Is demand elastic or inelastic at $q = 900$? For what values of q is demand elastic?

Since $|\eta| = \frac{10}{9} > 1$ $\boxed{\text{demand is elastic}}$ at $q = 900$

Since $|\eta| = \frac{q+100}{q} > 1$ for every q $\boxed{\text{demand is always elastic}}$

A

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3. *[17 marks]*

[6] (a) If $y = \dfrac{2^{x^2}(x-4)^{2x}}{e^{-x}(x^3+1)^4}$, find $\dfrac{dy}{dx}$ in terms of x only.

Easiest way: $\ln y = x^2 \ln 2 + 2x \ln(x-4) + x + 4\ln(x^3+1)$

$$\frac{1}{y}\frac{dy}{dx} = 2x\ln 2 + 2\ln(x-4) + \frac{2x}{x-4} + 1 + \frac{12x^2}{x^3+1}$$

$$\frac{dy}{dx} = \boxed{\frac{2^{x^2}(x-4)^{2x}}{e^{-x}(x^3+1)^4}\left[2x\ln 2 + 2\ln(x-4) + \frac{2x}{x-4} + 1 + \frac{12x^2}{x^3+1}\right]}$$

(b) (For part (b), simplify your answers as much as possible.)
If $y^3 = e^{x-y}$ defines y implicitly as a function of x,

[6] (i) find $\dfrac{dy}{dx}$ in terms of y **only**.

$$3y^2\frac{dy}{dx} = e^{x-y}\left(1-\frac{dy}{dx}\right)$$

$$\frac{dy}{dx}(3y^2 + e^{x-y}) = e^{x-y}$$

$$\frac{dy}{dx} = \frac{e^{x-y}}{3y^2+e^{x-y}} = \frac{y^3}{3y^2+y^3} = \boxed{\frac{y}{3+y}}$$

[5] (ii) With the same y, find $\dfrac{d^2y}{dx^2}$ in terms of y **only**.

$$\frac{d^2y}{dx^2} = \frac{(3+y)\frac{dy}{dx} - y\frac{dy}{dx}}{(3+y)^2} = \frac{3\frac{dy}{dx}}{(3+y)^2}$$

$$= \boxed{\frac{3y}{(3+y)^3}}$$

A

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4. *[16 marks]*

[8] (a) Recall that the relative rate of change of $y = f(x)$ is $\frac{1}{y}\frac{dy}{dx}$.

Given the total cost function $\qquad c = q(1+\frac{1}{q})^q,$

find the relative rate of change of **average** cost when $q = 10$. (Your answer should be a single number to 4 decimal places.)

relative rate of change $= \frac{1}{y}\frac{dy}{dx} = \frac{d}{dx}(\ln y)$

Av cost $\bar{c} = \frac{c}{q} = (1+\frac{1}{q})^q$

$$\frac{d}{dq}(\ln \bar{c}) = \frac{d}{dq}\left[q \ln(1+\frac{1}{q})\right]$$

$$= \ln(1+\frac{1}{q}) + \frac{q}{1+\frac{1}{q}} \cdot -\frac{1}{q^2}$$

$$= \ln(1+\frac{1}{q}) - \frac{\frac{1}{q}}{1+\frac{1}{q}}$$

$$= \ln 1.1 - \frac{1}{11} \approx \boxed{.0044}$$

[8] (b) If $f(x) = (2x)^{\ln x}$ when $x > 0$, find all critical points of $y = f(x)$ with $x > 0$. (Do **not** classify.)

$\ln f = \ln x \ln(2x)$

$$\frac{1}{f} f' = \frac{1}{x}\ln 2x + \frac{\ln x}{x}$$

$$= \frac{\ln 2x + \ln x}{x} = \frac{\ln 2 + \ln x + \ln x}{x} = \frac{\ln 2 + 2\ln x}{x}$$

$f' = (2x)^{\ln x}\frac{(\ln 2 + 2\ln x)}{x}$; For $x>0$, critical only when $f'=0$

i.e. when $\ln x = -\frac{1}{2}\ln 2 = \ln(\frac{1}{\sqrt{2}})$

i.e. $\boxed{x = \frac{1}{\sqrt{2}}}$