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Department of Mathematics
University of Toronto

Tuesday, January 15, 2013, 6:10 - 8 PM
MAT 133Y TERM TEST #2

Calculus and Linear Algebra for Commerce
Duration: 1 hour 50 minutes

Aids Allowed: A non-graphing calculator, with empty memory, to be supplied by student.

Instructions: Fill in the information on this page, and make sure your test booklet contains 10 pages. In addition, you should have a **multiple-choice answer sheet**, on which you should fill in your name, number, tutorial time, tutorial room, and tutor's name.

This test consists of 10 multiple choice questions, and 4 written-answer questions.
For the **multiple choice questions** you can do your rough work in the test booklet, but you must record your answer by circling the appropriate letter **on the answer sheet** with your pencil. Each correct answer is worth 4 marks; a question left blank, or an incorrect answer, or two answers for the same question is worth 0. For the **written-answer questions**, present your solutions in the space provided. The value of each written-answer question is indicated beside it.

ENCLOSE YOUR FINAL ANSWER IN A BOX AND WRITE IT IN INK.

TOTAL MARKS: 100

FAMILY NAME: ______________________

GIVEN NAME: ______________________

STUDENT NO: ______________________

SIGNATURE: ______________________

TUTORIAL TIME and ROOM: ______________________

REGCODE and TIMECODE: ______________________

T.A.'S NAME: ______________________

Regcode	Timecode	Room	**Regcode**	Timecode	Room
T0101A	M9A	SS1084	T0501C	W3C	UC328
T0101B	M9B	SS1086	T0501D	W3D	BA1180
T0201A	M3A	LM 155	T0601A	R4A	MP137
T0201B	M3B	LM 123	T0601B	R4B	MS2173
T0201C	M3C	RW142	T0601C	R4C	BA1220
T0201D	M3D	BA1180	T0601D	R4D	BA B024
T0301A	T3A	RW143	T0701A	F2A	SS1073
T0301B	T3B	WI524	T0701B	F2B	RW229
T0301C	T3C	MP134	T0701C	F2C	RW142
T0401A	W9A	SS1070	T0701D	F2D	MP134
T0401B	W9B	SS1072	T0801A	F3A	SS1087
T0401C	W9C	SS1074	T0801B	F3B	LM 157
T0501A	W3A	SS1087	T5101A	M5A	SS1069
T0501B	W3B	AB114	T5201A	M6A	LM 158

FOR MARKER ONLY	
Multiple Choice	
B1	
B2	
B3	
B4	
TOTAL	

A

NAME: ____________________ STUDENT NO: ____________

PART A. Multiple Choice

1. *[4 marks]*

If $f(x) = \dfrac{x^4}{(x^2+1)^2}$, then $f'(1) =$

A. 3

B. $\frac{1}{4}$

C. 1

D. $\frac{3}{4}$

E. $\frac{1}{2}$

$$f'(x) = \frac{(x^2+1)^2 \cdot 4x^3 - x^4 \cdot 2(x^2+1) \cdot 2x}{(x^2+1)^4}$$

$$f'(1) = \frac{2^2 \cdot 4 - 2 \cdot 2 \cdot 2}{2^4} = \frac{8}{2^4} = \boxed{\frac{1}{2}}$$

E

or: $\ln f = 4 \ln x - 2 \ln(x^2+1)$

$$\frac{1}{f} f' = \frac{4}{x} - \frac{4x}{x^2+1} = \frac{4}{1} - \frac{4}{2} = 2 \text{ at } x = 1$$

But $f(1) = \frac{1}{4}$

So $f'(1) = \frac{1}{4} \cdot 2 = \boxed{\frac{1}{2}}$ just as before

2. *[4 marks]*

If $p = 800 - q$ is the price when q units are sold and $q = \dfrac{200m - m^2}{50}$ is the quantity produced by m employees, then when $m = 50$ the marginal revenue product is:

A. 500

B. 1400

C. 150

D. 550

E. 1000

$$\frac{dr}{dm} = \frac{dr}{dq}\frac{dq}{dm}$$

$$r = pq = (800-q)q$$

$$\frac{dr}{dq} = 800 - 2q \qquad \frac{dq}{dm} = \frac{200-2m}{50}$$

when $m = 50$ $\quad q = \dfrac{200 \cdot 50 - 50^2}{50} = 150$

So $\dfrac{dr}{dq} = 800 - 2 \cdot 150 = 500$

$$\frac{dq}{dm} = \frac{200 - 2 \cdot 50}{50} = 2$$

$$\frac{dr}{dm} = 500 \times 2 = \boxed{1000}$$

E

NAME: ____________________ STUDENT NO: ____________________

3. *[4 marks]*

If $4S^2 + I^2 = 4SI + 2I - 8$ where S is savings and I is income, then when $S = 8$ and $I = 12$, the marginal propensity to consume is:

A. $\frac{3}{8}$

B. $-\frac{11}{8}$

C. $\frac{27}{32}$

D. $\frac{5}{8}$

E. $\frac{5}{32}$

$$\frac{dC}{dI} = 1 - \frac{dS}{dI}$$

$$8S\frac{dS}{dI} + 2I = 4\frac{dS}{dI}I + 4S + 2$$

$$64\frac{dS}{dI} + 24 = 4 \cdot \frac{dS}{dI} \cdot 12 + 32 + 2$$

$$16\frac{dS}{dI} = 10$$

$$\frac{dS}{dI} = \frac{5}{8}$$

$$\frac{dC}{dI} = \boxed{\frac{3}{8}}$$ A

4. *[4 marks]*

If $y(x)$ satisfies $(y+1)(y+2)(y+3) = x$, then when $(x, y) = (6, 0)$, $\frac{dy}{dx} =$

A. $\frac{1}{8}$

B. $\frac{1}{6}$

C. $\frac{1}{12}$

D. $\frac{1}{11}$

E. $\frac{1}{9}$

$$\frac{dy}{dx}(y+2)(y+3) + (y+1)\frac{dy}{dx}(y+3) + (y+1)(y+2)\frac{dy}{dx} = 1$$

When $y = 0$, $6\frac{dy}{dx} + 3\frac{dy}{dx} + 2\frac{dy}{dx} = 1$

$$\frac{dy}{dx} = \boxed{\frac{1}{11}}$$ D

or: $\ln(y+1) + \ln(y+2) + \ln(y+3) = \ln x$

$$\frac{1}{y+1}\frac{dy}{dx} + \frac{1}{y+2}\frac{dy}{dx} + \frac{1}{y+3}\frac{dy}{dx} = \frac{1}{x}$$

When $y=0$, $x=6$

$$\frac{dy}{dx} + \frac{1}{2}\frac{dy}{dx} + \frac{1}{3}\frac{dy}{dx} = \frac{1}{6}$$

$$\left(1 + \frac{1}{2} + \frac{1}{3}\right)\frac{dy}{dx} = \frac{1}{6}$$

$$\frac{dy}{dx} = \frac{\frac{1}{6}}{1 + \frac{1}{2} + \frac{1}{3}} = \frac{1}{6+3+2} = \boxed{\frac{1}{11}}$$

NAME: ____________________ STUDENT NO: ____________

5. *[4 marks]*

If the cost function C is given by

$$C(q) = e^{\frac{q^2}{16}}\sqrt{q^2+9}, \text{ for } 0 < q < 10,$$

then the derivative of the average cost function $\bar{C}(q) = \frac{C(q)}{q}$ at $q = 4$ is closest to

[Hint: logarithmic differentiation is probably the quickest way to do this problem]

A. 2.4

B. 3.4

C. 0.4

D. 1.4

E. 4.4

$\ln \bar{C} = \frac{q^2}{16} + \frac{1}{2}\ln(q^2+9) - \ln q$

$\frac{1}{\bar{C}}\frac{d\bar{C}}{dq} = \frac{q}{8} + \frac{q}{q^2+9} - \frac{1}{q}$

and $\bar{C}(4) = \frac{e\sqrt{25}}{4} = 5e/4$

So $\frac{d\bar{C}}{dq} = \frac{5e}{4}\left(\frac{1}{2} + \frac{4}{25} - \frac{1}{4}\right) = \frac{41e}{80}$

≈ 1.39

$\approx \boxed{1.4}$ D

6. *[4 marks]*

If $b > 0$ is fixed and $y = f(x) = b^x$, then at $x = 0$, $\frac{d^6y}{dx^6} =$

A. e^{6b}

B. $(\ln b)^6$

C. b^6

D. e^b

E. $6\ln(b)$

$f'(x) = b^x \ln b$

$f''(x) = b^x(\ln b)^2$

$f^{(6)}(x) = b^x(\ln b)^6$

$f^{(6)}(0) = b^0(\ln b)^6 = \boxed{(\ln b)^6}$ B

NAME: ______________________ STUDENT NO: ______________

7. *[4 marks]*

Using Newton's Method to approximate a value of x so that $e^x = 3 - x$, and starting at $x_1 = 0$, gives $x_3 =$

$$f(x) = e^x - 3 + x$$

$$f'(x) = e^x + 1$$

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} = x_n - \frac{e^{x_n} - 3 + x_n}{e^{x_n} + 1}$$

$$x_2 = 0 - \frac{e^0 - 3 + 0}{e^0 + 1} = 1$$

$$x_3 = 1 - \frac{e - 3 + 1}{e + 1} = 1 - \frac{e-2}{e+1} = \frac{3}{e+1}$$

$$= \boxed{.806824264} \quad C$$

A. 1

B. $\frac{1}{2}$

C. .806824264

D. .792134959

E. .79205997

8. *[4 marks]*

$\lim_{x \to 1} \frac{e^{x-1} - 1}{x^2 - 1}$

$$\frac{0}{0} \quad = \lim_{x \to 1} \frac{e^{x-1}}{2x} = \boxed{\frac{1}{2}} \quad C$$

A. $-\frac{1}{2}$

B. 0

C. $\frac{1}{2}$

D. 1

E. 2

NAME: ______________________ STUDENT NO: ______________

9. *[4 marks]*

$\lim_{x\to\infty} x^{\frac{1}{\ln x}} =$

A. e

B. 1

C. ∞

D. 0

E. $\frac{1}{e}$

Let $y = x^{\frac{1}{\ln x}}$

$\ln y = \frac{1}{\ln x} \ln x = 1$

So $\lim_{x\to\infty} \ln y = 1$

and $\lim_{x\to\infty} y = \boxed{e}$ A

10. *[4 marks]*

$\lim_{x\to 0} \frac{7^x - 3^x}{x} =$

A. $e^{7/3}$

B. $\ln\left(\frac{7}{3}\right)$

C. $\ln 10$

D. $\ln 4$

E. 1

$7^0 - 3^0 = 1 - 1 = 0$ so $\frac{0}{0}$

$\lim_{x\to 0} \frac{7^x \ln 7 - 3^x \ln 3}{1}$

$= \ln 7 - \ln 3 = \boxed{\ln\left(\frac{7}{3}\right)}$ B

A

NAME: ______________________ STUDENT NO: ______________

PART B. Written-Answer Questions

1. *[16 marks]*

Given

$$f(x) = \begin{cases} \dfrac{1-e^x}{x} & \text{if } x < 0 \\ -1 & \text{if } x = 0 \\ \dfrac{1-3^{1/x}}{1+3^{1/x}} & \text{if } 0 < x < 1 \\ \dfrac{|1-x|}{x-x^2} & \text{if } x \geq 1 \end{cases}$$

Evaluate the following limits: (show all steps)

[3] (a) $\lim_{x\to 0^+} f(x) = \lim_{x\to 0^+} \frac{1-3^{\frac{1}{x}}}{1+3^{\frac{1}{x}}} = \lim_{x\to 0^+} \frac{-3^{\frac{1}{x}}\ln 3\,(-\frac{1}{x^2})}{3^{\frac{1}{x}}\ln 3\,(-\frac{1}{x^2})}$ since $\frac{-\infty}{\infty}$

$= \boxed{-1}$

or $= \lim_{x\to 0^+} \frac{3^{-\frac{1}{x}}}{3^{-\frac{1}{x}}} \frac{(1-3^{\frac{1}{x}})}{(1+3^{\frac{1}{x}})} = \lim_{x\to 0^+} \frac{3^{-\frac{1}{x}}-1}{3^{-\frac{1}{x}}+1} = \boxed{-1}$

because $\lim_{x\to 0^+} 3^{-\frac{1}{x}} = 0$.

[3] (b) $\lim_{x\to 0^-} f(x)$

$= \lim_{x\to 0^-} \frac{1-e^x}{x} = \lim_{x\to 0^-} \frac{-e^x}{1}$ since $\frac{0}{0}$

$= \boxed{-1}$

[3] (c) $\lim_{x\to 1^+} f(x)$

$= \lim_{x\to 1^+} \frac{|1-x|}{x-x^2} = \lim_{x\to 1^+} \frac{x-1}{x(1-x)}$ since $x > 1$ $= \lim_{x\to 1^+} -\frac{1}{x} = \boxed{-1}$

[3] (d) $\lim_{x\to 1^-} f(x)$

$= \lim_{x\to 1^-} \frac{1-3^{\frac{1}{x}}}{1+3^{\frac{1}{x}}} = \frac{1-3}{1+3} = \boxed{-\frac{1}{2}}$

Determine whether f is continuous at each of the following values of x. Justify your answer.

[2] (e) $x = 0$ $\quad$ f is cont.

since $f(0) = -1$ and $\lim_{x\to 0^+} f(x) = \lim_{x\to 0^-} f(x) = f(0) = -1$

[2] (f) $x = 1$

f is not cont $\quad$ since either f is not defined at $x = 1$

or $\lim_{x\to 1^+} f(x) \neq \lim_{x\to 1^-} f(x)$

NAME: ______________________ STUDENT NO: ______________

2. *[14 marks]*

[8] (a) Find all values of x so that

$$\frac{e^x - e^2}{\ln x} \geq 0 \quad = 0 \text{ when } x=2$$

undefined when $x \leq 0$ and $x = 1$

$\frac{e^x - e^2}{\ln x}$ is cont and $\neq 0$ on

$(0,1)$	$x = .5$	$\ln x < 0$	$e^x - e^2 < 0$	so > 0
$(1,2)$	$x = 1.5$	$\ln x > 0$	$e^x - e^2 < 0$	so < 0
$(2,\infty)$	$x = 10^{100}$			> 0

The solution set is $\boxed{(0,1) \cup [2,\infty)}$ since $= 0$ at $x = 2$.

The sln set is $\boxed{0 < x < 1 \text{ or } 2 \leq x}$

[6] (b) How does the solution change if the inequality changes to

$$\frac{e^x - e^2}{\ln |x|} \geq 0 \ ?$$

For $x > 0$ nothing has changed.

But the expression is now also cont. (and $\neq 0$) on $(-\infty,-1)$ and $(-1,0)$. The numerator is always < 0 here

$(-1,0)$	$\ln\lvert x\rvert < 0$	so $\frac{e^x - e^2}{\ln\lvert x\rvert} > 0$
$(-\infty,-1)$	$\ln\lvert x\rvert > 0$	so $\frac{e^x - e^2}{\ln\lvert x\rvert} < 0$

The solution set now also includes $\boxed{(-1,0)}$ $-1 < x < 0$

i.e. $\boxed{(-1,0) \cup (0,1) \cup [2,\infty)}$

3. *[14 marks]*

The demand function for a product is given by

$$p = \sqrt{2500 - q^2}, \text{ for } 0 < q < 50$$

where q is quantity and p is unit price.

[4] (a) Find the marginal revenue when $q = 30$.

$p^2 + q^2 = 2500 \qquad 2p\frac{dp}{dq} + 2q = 0 \qquad \frac{dp}{dq} = -\frac{q}{p}$

$r = pq \quad \frac{dr}{dq} = q\frac{dp}{dq} + p$. When $q = 30$, $p = 40$

and $\frac{dp}{dq} = -\frac{30}{40}$, so $\frac{dr}{dq} = 30\left(-\frac{30}{40}\right) + 40 = \boxed{17.5}$

[4] (b) Find the elasticity of demand when $q = 30$.

$\eta = \frac{\frac{dq}{q}}{\frac{dp}{p}} = \frac{p}{q\frac{dp}{dq}} = \frac{p}{q\left(-\frac{q}{p}\right)} = -\frac{p^2}{q^2} = -\frac{40^2}{30^2} = \frac{-16}{9}$

$\boxed{\eta = \frac{-16}{9} \approx -1.78}$

[3] (c) For what values of p and q does demand have unit elasticity?

$\eta = -1$ when $p^2 = q^2$

$2q^2 = 2500$

$p^2 = q^2 = 1250$

$\boxed{p = q = 25\sqrt{2} \approx 35.36}$

[3] (d) For what values of p and q is demand elastic?

$|\eta| > 1 \qquad p^2 > q^2 \qquad 2500 - q^2 > q^2 \qquad 2500 > 2q^2$

$1250 > q^2$

$25\sqrt{2} > q$

and $p^2 > 2500 - p^2$

$p^2 > 1250 \qquad p > 25\sqrt{2}$

so $\boxed{q < 25\sqrt{2} < p}$

NAME: ____________________ STUDENT NO: ____________

4. *[16 marks]*

Suppose $y(x)$ satisfies

$$x^2 + 2xy + 5y^2 = 8$$

[8] (a) Find $\dfrac{dy}{dx}$ in terms of x and y.

$$2x + 2y + 2x\frac{dy}{dx} + 10y\frac{dy}{dx} = 0$$

$$(2x + 10y)\frac{dy}{dx} = -(2x + 2y)$$

$$\boxed{\frac{dy}{dx} = -\left(\frac{x+y}{x+5y}\right)}$$

[8] (b) Find $\dfrac{d^2y}{dx^2}$ when $x = 1$ and $y = 1$. At $(1,1)$ $\frac{dy}{dx} = -\frac{2}{6} = -\frac{1}{3}$

Method 1 : Quotient Rule

$$\frac{d^2y}{dx^2} = -\left[\frac{(x+5y)\left(1+\frac{dy}{dx}\right) - (x+y)\left(1+5\frac{dy}{dx}\right)}{(x+5y)^2}\right]$$

$$= -\left[\frac{6\left(1-\frac{1}{3}\right) - 2\left(1-\frac{5}{3}\right)}{36}\right] = -\left[\frac{4+\frac{4}{3}}{36}\right]$$

$$= \boxed{-\frac{4}{27}}$$

Method 2: $(x+5y)\frac{dy}{dx} = -x-y$

$$\left(1+5\frac{dy}{dx}\right)\frac{dy}{dx} + (x+5y)\frac{d^2y}{dx^2} = -1 - \frac{dx}{dx}$$

$$\left(1-\frac{5}{3}\right)\left(-\frac{1}{3}\right) + (1+5)\frac{dy^2}{dx^2} = -1 + \frac{1}{3}$$

$$\frac{2}{9} + 6\frac{d^2y}{dx^2} = -\frac{2}{3} \qquad 6\frac{d^2y}{dx^2} = -\frac{8}{9} \qquad \boxed{\frac{d^2y}{dx^2} = -\frac{4}{27}}$$