Solns

A

Department of Mathematics
University of Toronto

Tuesday, January 17, 2012, 6:10 - 8:00 PM
MAT 133Y TERM TEST #2

Calculus and Linear Algebra for Commerce
Duration: 1 hour 50 minutes

Aids Allowed: A non-graphing calculator, with empty memory, to be supplied by student.

Instructions: Fill in the information on this page, and make sure your test booklet contains 10 pages. In addition, you should have a **multiple-choice answer sheet**, on which you should fill in your name, number, tutorial time, tutorial room, and tutor's name.

This test consists of 10 multiple choice questions, and 4 written-answer questions.
For the **multiple choice questions** you can do your rough work in the test booklet, but you must record your answer by circling the appropriate letter **on the answer sheet** with your pencil. Each correct answer is worth 4 marks; a question left blank, or an incorrect answer, or two answers for the same question is worth 0. For the **written-answer questions**, present your solutions in the space provided. The value of each written-answer question is indicated beside it.

ENCLOSE YOUR FINAL ANSWER IN A BOX AND WRITE IT IN INK.

TOTAL MARKS: 100

FAMILY NAME: _____

GIVEN NAME: _____

STUDENT NO: _____

SIGNATURE: _____

TUTORIAL TIME and ROOM: _____

REGCODE and TIMECODE: _____

T.A.'S NAME: _____

Regcode	Timecode	Room	Regcode	Timecode	Room
T0101A	M9A	SS1072	T0601A	R4A	RW142
T0101B	M9B	SS1073	T0601B	R4B	GB248
T0101C	M9C	SS1083	T0601C	R4C	AB107
T0201A	M3A	SS2106	T0601D	R4D	GB221
T0201B	M3B	MP134	T0701A	F2A	LM155
T0201C	M3C	RW143	T0701B	F2B	RW229
T0201D	M3D	UC328	T0701C	F2C	BA1240
T0301A	T3A	RW229	T0701D	F2D	MS4279
T0301B	T3B	WI524	T0801A	F3A	SS1085
T0301C	T3C	MP137	T0801B	F3B	SS2110
T0401A	W9A	SS1072	T0801C	F3C	WI 523
T0401B	W9B	SS1073	T5101A	M5A	SS1087
T0501A	W3A	SS1070	T5101B	M5B	MP134
T0501B	W3B	WI523	T5101C	M5C	MP137
			T5201A	M6A	LM162

FOR MARKER ONLY	
Multiple Choice	
B1	
B2	
B3	
B4	
TOTAL	

NAME: _____ STUDENT NO: _____

PART A. Multiple Choice

1. *[4 marks]*

If $\frac{e^x-1}{x^2+x-2} \leq 0$ then

A. $x < -2$ or $0 < x < 1$

B. $-2 < x \leq 0$ or $x > 1$

C. $x < -2$ or $0 \leq x < 1$

D. $x \leq -2$ or $0 \leq x \leq 1$

E. $-2 < x < 0$ or $x > 1$

$\frac{e^x-1}{x^2+x-2} = \frac{e^x-1}{(x+2)(x-1)}$

$= 0$ at $x = 0$

undefined at $x = 1$ and $x = -2$

Cont all the rest of the time.

$(-\infty, -2)$ try $x = -1000$: all factors < 0 so < 0.

$(-2, 0)$ try $x = -1$: $e^x - 1 < 0$ and $x - 1 < 0$ so > 0

$(0, 1)$ $e^x - 1 > 0$ and $x + 2 > 0$ so < 0

$(1, \infty)$ > 0

Ans $(-\infty, -2) \cup [0, 1)$ (C)

2. *[4 marks]*

If $s = \frac{t}{t+\frac{1}{1+t}}$, then when $t = 1$, $\frac{ds}{dt} =$

A. $-16/9$

B. $-10/9$

C. $2/3$

D. $1/3$

E. $-1/3$

$$= \frac{\left(t + \frac{1}{1+t}\right) - t\left[1 - \frac{1}{(1+t)^2}\right]}{\left(t + \frac{1}{1+t}\right)^2}$$

$$= \frac{\left(1 + \frac{1}{2}\right) - \left(1 - \frac{1}{4}\right)}{\left(1 + \frac{1}{2}\right)^2}$$

$$= \frac{\frac{3}{2} - \frac{3}{4}}{\frac{9}{4}} = \frac{\frac{3}{4}}{\frac{9}{4}} = \frac{1}{3}$$ (D)

NAME: _____ STUDENT NO: _____

3. *[4 marks]*

If $y = x\ln(x-2.5)$, then $y'(3) =$

A. $2+\ln(0.5)$

B. $3\ln(0.5)$

C. $3+\ln(0.5)$

D. $0.5+\ln(0.5)$

E. $6+\ln(0.5)$

$y' = \ln(x-2.5) + \frac{x}{x-2.5}$

$= \ln(.5) + \frac{3}{.5}$ at $x=3$

$= \boxed{6+\ln(.5)}$ (E)

4. *[4 marks]*

If $f(x) = \frac{x^2+4e^{(x^2)}}{e^{\sqrt{x}}}$, then $f'(4)$ equals

A. $\frac{4+3e^{16}}{e^2}$

B. $\frac{4-3e^{16}}{e^2}$

C. $\frac{4+31e^{16}}{e^2}$

D. $\frac{-8+4e^{16}}{e^2}$

E. $\frac{-8+28e^{16}}{e^2}$

$f'(x) = \frac{e^{\sqrt{x}}(2x+4e^{x^2}\cdot 2x) - (x^2+4e^{x^2})e^{\sqrt{x}}\frac{}{2\sqrt{x}}}{(e^{\sqrt{x}})^2}$

$f'(4) = \frac{e^2(8+32e^{16}) - (16+4e^{16})e^2}{(e^2)^2}$

$= \frac{8+32e^{16}-4-e^{16}}{e^2}$

$= \boxed{\frac{4+31e^{16}}{e^2}}$ (C)

NAME: _____ STUDENT NO: _____

5. *[4 marks]*

The relationship between national consumption, C, and national income, I, is given by

$$C^2 + CI - I^2 + 400 = 0$$

where C and I are in billions of dollars. What is the marginal propensity to **save** when $I = 100$ and $C = 60$ billion dollars?

A. .2

B. .36

C. .64

D. .8

E. 1.57

$$2C\frac{dC}{dI} + C + I\frac{dC}{dI} - 2I = 0$$

$$120\frac{dC}{dI} + 60 + 100\frac{dC}{dI} - 200 = 0$$

$$220\frac{dC}{dI} = 140$$

$$\frac{dC}{dI} = \frac{7}{11}$$

$$\frac{dS}{dI} = 1 - \frac{dC}{dI} = \frac{4}{11} \approx \boxed{.36} \quad (B)$$

6. *[4 marks]*

The relative rate of change of y when $y = (x+1)^{x^2-1}$ and $x = 2$ is

A. .11

B. 5.39

C. 9

D. 36

E. 145.65

We are looking for $\frac{1}{y}\frac{dy}{dx}$ at $x = 2$

$$\ln y = (x^2-1)\ln(x+1)$$

$$\frac{1}{y}\frac{dy}{dx} = \frac{x^2-1}{x+1} + 2x\ln(x+1)$$

at $x=2$ $\quad = 1 + 4\ln 3 \approx \boxed{5.39}$ (B)

NAME: _____ STUDENT NO: _____

7. *[4 marks]*

If $R(x) = f(x)g[h(x)]$ and $f(2) = -2,\ g(2) = -1,\ h(2) = 2,\ f'(2) = 3,$
$g'(2) = -3,\ h'(2) = -2,$ then $R'(2) =$

A. 5

B. 18

C. 3

D. -15

E. -9

$R'(x) = f'(x)g[h(x)] + f(x)g'(h(x))h'(x)$

$R'(2) = f'(2)g(h(2)) + f(2)g'(h(2))h'(2)$

$= 3g(2) + (-2)g'(2)(-2)$

$= 3g(2) + 4g'(2)$

$= -3 - 12 = \boxed{-15}$ (D)

8. *[4 marks]*

$\lim_{x\to\infty} \dfrac{5x^4 - 3x^3 + 7x^2 + 2x - 3}{4x^4 + 5x^3 - 6x^2 + x - 7} = \boxed{\dfrac{5}{4}}$ (E)

A. does not exist

B. $= \frac{3}{7}$

C. $= 5(4!)$

D. $= \frac{5}{4(4!)}$

E. $= \frac{5}{4}$

for example by

$$\frac{x^4\left(5 - \frac{3}{x^2} + \frac{7}{x^2} + \frac{2}{x^3} - \frac{3}{x^4}\right)}{x^4\left(4 + \frac{5}{x} - \frac{6}{x^2} + \frac{1}{x} - \frac{7}{x^4}\right)} \to \frac{5}{4}$$

NAME: _____ STUDENT NO: _____

9. *[4 marks]*

$\lim_{x\to 2} \dfrac{2^x - 4}{3^x - 9}$

$\frac{0}{0}$ so by L'Hôpital

$\lim = \lim_{x\to 2} \dfrac{2^x \ln 2}{3^x \ln 3} = \boxed{\dfrac{4\ln 2}{9\ln 3}}$ (A)

A. $= \dfrac{4\ln 2}{9\ln 3}$

B. $= \dfrac{4}{9}$

C. $= \dfrac{\ln 2}{\ln 3}$

D. $= \dfrac{9\ln 3}{4\ln 2}$

E. does not exist

10. *[4 marks]*

If two iterations of Newton's method are used to estimate a root of $f(x) = x^3 + x + 1$, beginning with the initial estimate $x_1 = 0$, then the third estimate, x_3, is closest to

A. -0.686

B. -1.000

C. -0.682

D. -0.750

E. -0.720

$x_{n+1} = x_n - \dfrac{f(x_n)}{f'(x_n)}$

$= x_n - \dfrac{x_n^3 + x_n + 1}{3x_n^2 + 1}$

$x_0 = 0$

$x_1 = 0 - \frac{1}{1} = -1$

$x_2 = -1 - \dfrac{(-1 - 1 + 1)}{3 + 1}$

$= -1 - (-\frac{1}{4}) = \boxed{-\frac{3}{4}}$ (D)

NAME: _____ STUDENT NO: _____

PART B. Written-Answer Questions

1. *[14 marks]*

Let $f(x) = xe^x$.

[3] (a) Compute $f'(x)$.

$f'(x) = e^x + xe^x = (1+x)e^x$

[3] (b) Compute $f''(x)$.

$f''(x) = e^x + e^x + xe^x = (2+x)e^x$

[8] (c) Find the equation of the tangent line to $y = f'(x)$ at $x = 1$. [Note: the tangent line to $y = f'(x)$, **not** to $y = f(x)$]

At $x=1$ $y = f'(1) = 2e$

and the slope is $f''(1) = 3e$

So the equation of the line is

$y - 2e = 3e(x-1)$

or $y = 3ex - e$

NAME: _____ STUDENT NO: _____

2. *[15 marks]*

If $y^3 + xy = x^3 + 1$:

[7] (a) Find $\frac{dy}{dx}$ in terms of x and y.

$$3y^2\frac{dy}{dx} + y + x\frac{dy}{dx} = 3x^2$$

$$\boxed{\frac{dy}{dx} = \frac{3x^2 - y}{3y^2 + x}}$$

Note that at $x=1, y=1$, $\frac{dy}{dx} = \frac{1}{2}$

We need this in part (b)

[8] (b) Find $\frac{d^2y}{dx^2}$ when $x = 1$ and $y = 1$.

2 ways: A: $\frac{d^2y}{dx^2} = \frac{d}{dx}\left[\frac{3x^2 - y}{3y^2 + x}\right]$

$$= \frac{(3y^2+x)(6x - \frac{dy}{dx}) - (3x^2 - y)(6y\frac{dy}{dx} + 1)}{(3y^2+x)^2}$$

$$= \frac{4(6 - \frac{1}{2}) - 2(3+1)}{16} = \frac{14}{16} = \boxed{\frac{7}{8}}$$

B: Starting from the first line in (a)

$$6y\left(\frac{dy}{dx}\right)^2 + 3y^2\frac{d^2y}{dx^2} + \frac{dy}{dx} + \frac{dy}{dx} + x\frac{d^2y}{dx^2} = 6x$$

$$6\left(\frac{1}{2}\right)^2 + 3\frac{d^2y}{dx^2} + \frac{1}{2} + \frac{1}{2} + \frac{d^2y}{dx^2} = 6$$

$$4\frac{d^2y}{dx^2} = \frac{14}{4}$$

$$\frac{d^2y}{dx^2} = \frac{14}{16} = \boxed{\frac{7}{8}} \text{ as before}$$

NAME: _____ STUDENT NO: _____

3. *[15 marks]*

The demand equation for a certain product is $q = 400 - p^2$.

[5] (a) Find the percentage rate of change of revenue with respect to p when p is 10.

$r = pq = p(400 - p^2)$

We are looking for $\frac{1}{r}\frac{dr}{dp}$ as a % at $p = 10$

$\frac{1}{r}\frac{dr}{dp} = \frac{1}{p(400-p^2)} \cdot (400 - 3p^2) = \frac{1}{10 \cdot 300} \cdot 100 = \frac{1}{30}$

$\approx$ .033 or 3.33%

[5] (b) Find the revenue function and the marginal revenue function in terms of q.

$p^2 = 400 - q \quad p = \sqrt{400-q}$

$r = pq = q\sqrt{400-q}$ revenue as a fcn of q

$\frac{dr}{dq} = \sqrt{400-q} + \frac{q(-1)}{2\sqrt{400-q}} = \frac{800-3q}{2\sqrt{400-q}}$

[5] (c) If $q = 35m - m^2$ where m is the number of employees required to produce q units of product, find the marginal revenue product when the number of employees is 20.

$\frac{dr}{dm} = \frac{dr}{dq}\frac{dq}{dm} = \frac{800-3q}{2\sqrt{400-q}}(35 - 2m)$

If $m = 20$, $q = 300$

so $\frac{dr}{dm} = -\frac{100}{2 \cdot \sqrt{100}}(-5) = 25$

NAME: _____ STUDENT NO: _____

4. *[16 marks]*

The demand function $p(q)$ for a certain product satisfies $q^2 + 2qp + 4p^2 = 108$ where $p > 0$ and $q > 0$ respectively denote unit price and quantity sold.

[5] (a) Find $\frac{dp}{dq}$ in terms of p and q.

$$2q + 2p + 2q\frac{dp}{dq} + 8p\frac{dp}{dq} = 0$$

$$\frac{dp}{dq} = -\frac{p+q}{4p+q}$$

[5] (b) Find the point elasticity of demand in terms of p and q.

$$\eta = \frac{\frac{dq}{q}}{\frac{dp}{p}} = \frac{p}{q}\frac{dp}{dq} = -\frac{p}{q}\frac{1}{\left(\frac{p+q}{4p+q}\right)} = \boxed{-\frac{p(4p+q)}{q(p+q)}}$$

[6] (c) For which p and q does demand have unit elasticity? $\eta = -1$

$$\frac{p(4p+q)}{q(p+q)} = 1 \quad \text{so} \quad 4p^2 + pq = pq + q^2$$

$$4p^2 = q^2$$

$$2p = q \quad (-2p = q \text{ makes no sense since } p, q > 0)$$

Going back to

$$q^2 + 2qp + 4p^2 = 108,$$

$$4p^2 + 4p^2 + 4p^2 = 108 \Rightarrow p^2 = 9$$

so $\boxed{p = 3 \text{ and } q = 6}$