Solved

Department of Mathematics
University of Toronto

MONDAY, December 1, 2008 6:10-8:00 PM

MAT 133Y TERM TEST #2

Calculus and Linear Algebra for Commerce
Duration: 1 hour 50 minutes

Aids Allowed: A non-graphing calculator, with empty memory, to be supplied by student.

Instructions: Fill in the information on this page, and make sure your test booklet contains 10 pages. In addition, you should have a **multiple-choice answer sheet**, on which you should fill in your name, number, tutorial time, tutorial room, and tutor's name.

This test consists of 10 multiple choice questions, and 4 written-answer questions.
For the **multiple choice questions** you can do your rough work in the test booklet, but you must record your answer by circling the appropriate letter **on the answer sheet** with your pencil. Each correct answer is worth 4 marks; a question left blank, or an incorrect answer, or two answers for the same question is worth 0. For the **written-answer questions**, present your solutions in the space provided. The value of each written-answer question is indicated beside it. ENCLOSE YOUR FINAL ANSWER IN A BOX AND WRITE IT IN INK.

TOTAL MARKS: 100

FAMILY NAME: ______________________

GIVEN NAME: ______________________

STUDENT NO: ______________________

SIGNATURE: ______________________

TUTORIAL TIME and ROOM: ______________________

REGCODE and TIMECODE: ______________________

T.A.'S NAME: ______________________

Regcode	Timecode	Room	**Regcode**	Timecode	Room
T0101A	M9A	SS1074	T0501D	W3D	BA2139
T0101B	M9B	SS2105	T0601A	R4A	LM 157
T0101C	M9C	SS2111	T0601B	R4B	LM 123
T0101D	M9D	LM 158	T0701A	F2A	RW 229
T0201A	M3A	RW 229	T0701B	F2B	SS2111
T0201B	M3B	LM 157	T0701C	F2C	SS2128
T0201C	M3C	RW 142	T0801A	F3A	LM 155
T0201D	M3D	UC 52	T0801B	F3B	LM 123
T0301A	T3A	RW 143	T5101A	M5A	MP 134
T0301B	T3B	MP 134	T5101B	M5B	SS2111
T0401A	W9A	SS1074	T5101C	M5C	MP 118
T0401B	W9B	SS1086	T5101D	M5D	RW 143
T0401C	W9C	LM 158	T5201A	M6A	LM 162
T0501A	W3A	SS2105			
T0501B	W3B	MS2173			
T0501C	W3C	UC 256			

FOR MARKER ONLY	
Multiple Choice	
B1	
B2	
B3	
B4	
TOTAL	

NAME: ______________________ STUDENT NO: ______________

PART A. Multiple Choice

1. *[4 marks]*

$$\lim_{x\to -1} \frac{2x^2-6x-8}{x^2-1} = \lim_{x\to -1} \frac{2(x-4)\cancel{(x+1)}}{(x-1)\cancel{(x+1)}} = \frac{2(-5)}{-2} = 5$$

A. is 2

B. is 0

C. is 5

D. is 1

E. does not exist

2. *[4 marks]*

$$\lim_{x\to\infty} \frac{(4-x)(1+2x)(1+x)}{9x-x^3} = \lim_{x\to\infty} \frac{\cancel{x^3}\left(\frac{4}{x}-1\right)\left(\frac{1}{x}+2\right)\left(\frac{1}{x}+1\right)}{\cancel{x^3}\left(\frac{9}{x^2}-1\right)}$$

$$= \frac{(-1)(2)(1)}{-1} = 2$$

A. 2

B. 0

C. -1

D. 1

E. $+\infty$

NAME: ______________________ STUDENT NO: ______________________

3. *[4 marks]*

$$\lim_{x\to-\infty} \frac{2e^{-x}+1}{1-e^{-x}} = \lim_{x\to\infty} \frac{\cancel{e^{-x}}(2+e^{x})}{\cancel{e^{-x}}(e^{x}-1)} = \frac{2}{-1} = -2$$

but $\lim_{x\to-\infty} e^{x} = 0$

A. 2

B. −1

C. −2

D. 1

E. $-\infty$

4. *[4 marks]*
All solutions of the inequality

$$\frac{e^x-2}{x^2(x+1)} \geq 0$$

are given by

A. $x < -1$ or $x > \ln 2$

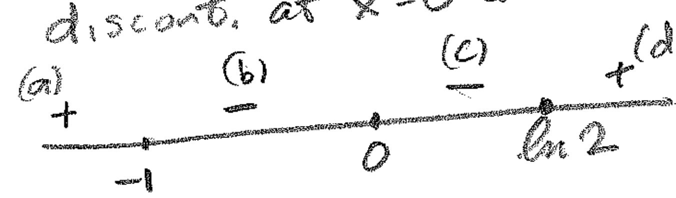
B. $-1 < x \leq \ln 2$

C. $-1 < x < 0$ or $x \geq \ln 2$

D. $x < -1$ or $x \geq \ln 2$

E. $x < -1$ or $0 < x \leq \ln 2$

0 at $x = \ln 2 > 0$.
discont. at $x = 0$ and $x = -1$



(a) For large negative x, $e^x \approx 0$
so $e^x - 2 < 0$, $x^2 > 0$, $x+1 < 0$

(b) $(-1, 0)$ say $x = -\frac{1}{2}$
$e^{-\frac{1}{2}} < 1$ so $e^x - 2 < 0$, but the denominator > 0

(c) $(0, \ln 2)$ say $x = \frac{1}{2}$
$e^{\frac{1}{2}} < 2$ so same as (b)

(d) For large positive x, everything is positive.

Answer:
$(-\infty, -1) \cup [\ln 2, \infty)$
otherwise known as

$\boxed{x < -1 \text{ or } x \geqslant \ln 2}$

NAME: ______________________ STUDENT NO: ______________

5. *[4 marks]*

$$\frac{d}{dx}\left(\frac{x}{x^2+1}e^x\right) =$$

A. $\dfrac{x^3-x^2+x+1}{(x^2+1)^2}\,e^x$

B. $\dfrac{x^3+x^2+x-1}{(x^2+1)^2}\,e^x$

C. $\dfrac{x^3-x^2-x+1}{(x^2+1)^2}\,e^x$

D. $\dfrac{x^3+x^2-x+1}{(x^2+1)^2}\,e^x$

E. $\dfrac{x^3-x^2+x-1}{(x^2+1)^2}\,e^x$

$$\frac{x}{x^2+1}e^x + e^x\,\frac{(x^2+1)-x\cdot 2x}{(x^2+1)^2}$$

$$= \frac{e^x}{(x^2+1)^2}\left[x(x^2+1)+(-x^2+1)\right]$$

$$= \frac{e^x\left[x^3-x^2+x+1\right]}{(x^2+1)^2}$$

6. *[4 marks]*

Let $f(x) = \dfrac{(4-x)^2(x+4)^{1/2}}{(x+1)(2-x)(3x+6)}$

Then $f'(0) =$

A. $-\dfrac{11}{3}$

B. $\dfrac{11}{3}$

C. $-\dfrac{8}{3}$

D. $\dfrac{8}{3}$

E. $-\dfrac{11}{8}$

$$\ln f = 2\ln(4-x)+\frac{1}{2}\ln(x+4)-\ln(x+1) - \ln(2-x) - \ln(3x+6)$$

$$\frac{1}{f}f' = \frac{2(-1)}{4-x}+\frac{1}{2(x+4)}-\frac{1}{x+1}-\frac{1(-1)}{2-x}-\frac{3}{3x+6}$$

$$f'(0) = f(0)\left[-\frac{2}{4}+\frac{1}{8}-1+\frac{1}{2}-\frac{3}{6}\right]$$

$$= f(0)\left(-\frac{11}{8}\right)$$

But $f(0) = \dfrac{4^2\cdot 4^{\frac{1}{2}}}{1\cdot 2\cdot 6} = \dfrac{16}{6} = \dfrac{8}{3}$

$$f'(0) = \frac{8}{3}\left(-\frac{11}{8}\right) = \boxed{-\frac{11}{3}}$$

NAME: ____________________ STUDENT NO: ____________________

7. *[4 marks]*

Let

$$y = (x+1)^{x^2-1}$$

Then $y' =$

A. $2x(x+1)(x^2-1)$

B. $(x^2-1)(x+1)^{x^2-2}$

C. $\left(2x\ln(x+1) + \frac{2x}{x+1}\right)\left((x+1)^{x^2-1}\right)$

D. $(x+1)^{x^2-1}\left(2x\ln(x+1) + (x-1)\right)$

E. $(x+1)^{x^2-1}\left(\ln(x^2-1) + 2x\ln(x+1)\right)$

$\ln y = (x^2-1)\ln(x+1)$

$\frac{1}{y}y' = 2x\ln(x+1) + \frac{x^2-1}{x+1}$

$\frac{1}{y}y' = 2x\ln(x+1) + (x-1)$

$y' = (x+1)^{x^2-1}\left[2x\ln(x+1) + (x-1)\right]$

8. *[4 marks]*

Let $2x^3 + 5xy + y^3 = 8$

Then when $x = 1$ and $y = 1$, $y' =$

A. $-\frac{11}{8}$

B. $-\frac{11}{6}$

C. 0

D. $-\frac{6}{11}$

E. $-\frac{8}{11}$

$6x^2 + 5y + 5xy' + 3y^2y' = 0$

At $(1,1)$

$6 + 5 + 5y' + 3y' = 0$

$8y' = -11$

$y' = -\frac{11}{8}$

or: Solve for y'

$y'(5x+3y^2) = -(6x^2+5y)$

$y' = -\frac{(6x^2+5y)}{5x+3y^2}$

$= -\frac{11}{8}$ at $x=1, y=1$

NAME: ______________________ STUDENT NO: ______________

9. *[4 marks]*

The point elasticity of the demand equation $q = p^2 - 10p + 50$ when $p = 3$ is

A. $\frac{3}{116}$

B. $-\frac{12}{29}$

C. $-\frac{3}{116}$

D. $\frac{6}{116}$

E. $-\frac{6}{29}$

$\mu = \frac{p}{q}\frac{dq}{dp} = \frac{p}{q\frac{dp}{dq}}$. Easier to use $\frac{dq}{dp} = 2p - 10$

when $p = 3$, $\frac{dq}{dp} = 6 - 10 = -4$

and $q = 9 - 30 + 50 = 29$

So $\mu = \frac{3}{29}(-4) = \frac{-12}{29}$

10. *[4 marks]*

If a company can produce $q(m) = 2m + \frac{64}{m}$ units when it has $m > 5$ employees and must set its price p at $p = \frac{48}{\sqrt{q}}$ to sell all q units, what is its marginal revenue product ($\frac{dr}{dm}$, where r denotes revenue) when $m = 16$?

A. 12

B. 7

C. 10

D. 6

E. 9

$\frac{dr}{dm} = \frac{dr}{dq}\frac{dq}{dm}$

$r = pq = \frac{48}{\sqrt{q}} q = 48\sqrt{q}$

$\frac{dr}{dq} = \frac{24}{\sqrt{q}}$ and $q(16) = 2 \cdot 16 + \frac{64}{16} = 36$

So $\frac{dr}{dq} = \frac{24}{\sqrt{36}} = 4$ when $m = 16$

and $\frac{dq}{dm} = 2 - \frac{64}{m^2} = 2 - \frac{64}{256} = 2 - \frac{1}{4} = \frac{7}{4}$

$\frac{dr}{dm} = 4 \times \frac{7}{4} = 7$

NAME: ______________________ STUDENT NO: ______________

PART B. Written-Answer Questions

1. *[15 marks]*

Given

$$f(x) = \begin{cases} \dfrac{|x|}{x^2+x} & \text{if } x<0 \\ -1 & \text{if } x=0 \\ \dfrac{1-2^{\frac{1}{x}}}{2+2^{\frac{1}{x}}} & \text{if } 0<x\le 1 \\ \dfrac{\sqrt{x+3}-2}{x-1} & \text{if } x>1. \end{cases}$$

Determine whether f is continuous at each of the values of x below. Show all steps and state clearly why f is or is not continuous.

[8] (a) $x=0$ $\quad f(0)=-1$ and since $x<0$

$\lim\limits_{x\to 0^-} f(x) = \lim\limits_{x\to 0^-} \frac{-x}{x^2+x} = \lim\limits_{x\to 0^-} \frac{-1}{x+1} = -1$ so far, so good.

$\lim\limits_{x\to 0^+} f(x) = \lim\limits_{x\to 0^+} \frac{2^{-\frac{1}{x}}-1}{2^{1-\frac{1}{x}}+1}$ but $-\frac{1}{x}\to-\infty$ so $2^{-\frac{1}{x}}\to 0$ and $2\cdot 2^{-\frac{1}{x}}\to 0$

Hence $\lim\limits_{x\to 0^+} f(x) = -1$ also.

Conclusion: $\lim\limits_{x\to 0} f(x) = -1 = f(0)$ and $\boxed{f \text{ is cont. at } x=0}$

[7] (b) $x=1$ $\quad f(1) = \frac{1-2}{2+2} = -\frac{1}{4}$

and $\lim\limits_{x\to 1^-} f(x) = \lim\limits_{x\to 1^-} \frac{1-2^{\frac{1}{x}}}{2+2^{\frac{1}{x}}} = -\frac{1}{4}$ without difficulty.

$\lim\limits_{x\to 1^+} f(x) = \lim\limits_{x\to 1^+} \frac{\sqrt{x+3}-2}{x-1}$

$= \lim\limits_{x\to 1^+} \frac{\cancel{x+3-4}}{\cancel{(x-1)}(\sqrt{x+3}+2)} = \frac{1}{\sqrt{4}+2} = \frac{1}{4}$

Since $\lim\limits_{x\to 1^+} f(x) \neq \lim\limits_{x\to 1^-} f(x)$, $\lim\limits_{x\to 1} f(x)$ does not exist

and $\boxed{f \text{ is not cont at } x=1}$

NAME: ______________________ STUDENT NO: ______________

2. *[15 marks]*

Let $y = \dfrac{4x}{\sqrt{x+1}}$

[10] (a) Find y' and y'' (expressed in as simple an algebraic form as possible.)

$$y' = \frac{\sqrt{x+1}\cdot 4 - 4x\cdot\frac{1}{2\sqrt{x+1}}}{x+1} = \frac{4(x+1)-2x}{(x+1)^{3/2}}$$

$$\boxed{y' = \frac{2x+4}{(x+1)^{3/2}}} \quad \text{or} \quad \frac{2(x+2)}{(x+1)^{3/2}}$$

$$y'' = \frac{(x+1)^{3/2}\cdot 2 - 2(x+2)\frac{3}{2}(x+1)^{\frac{1}{2}}}{(x+1)^3}$$

$$= \frac{2(x+1)-3(x+2)}{(x+1)^{5/2}}$$

$$\boxed{y'' = -\frac{x+4}{(x+1)^{5/2}}}$$

You can get here by the product rule used on $y = 4x(x+1)^{-\frac{1}{2}}$ or even by logarithmic differentiation.

[5] (b) Find the equation of the tangent line to the graph of y' at the point $\left(3, \frac{10}{8}\right)$

[Note: the graph of y', **not** the graph of y]

Note ~~that~~ when $x = 3$, $y' = \dfrac{2\cdot 3+4}{4^{3/2}} = \dfrac{10}{8}$

The slope of the tan line to y' at $(3, \frac{10}{8})$ is $y''(3) = -\dfrac{7}{4^{5/2}} = -\dfrac{7}{32}$

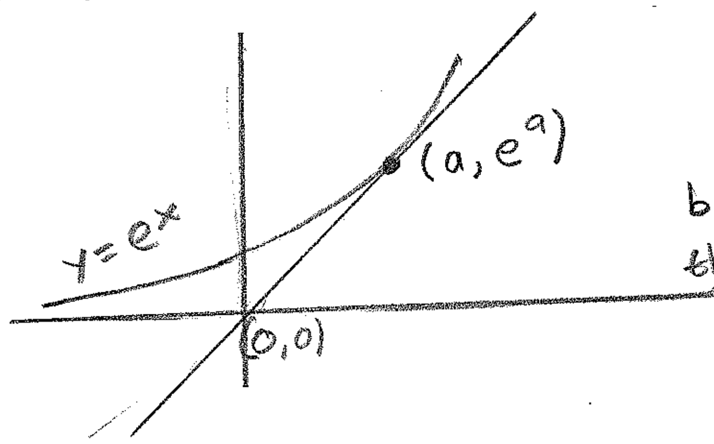
So the equation of the tan line is

$$\boxed{y - \frac{10}{8} = -\frac{7}{32}(x-3)} \quad \text{or} \quad y = -\frac{7}{32}x + \frac{61}{32}$$

NAME: ____________________ STUDENT NO: ____________________

3. *[15 marks]*

Find the equation of the line in the xy-plane which goes through the origin and is tangent to the graph of $y = e^x$. **Show your work.**



$y' = e^x$

Let $x=a, y=e^a$ be the point where the line and the curve intersect, tangent to each other.

The slope of the line is e^a

The equation of the line is

$$y-e^a = e^a(x-a)$$

The problem is, we need to find a:

The slope of the line is

$$\frac{e^a-0}{a-0} = \frac{e^a}{a} \quad \text{and also } e^a$$

So $\frac{e^a}{a} = e^a$ which means $a=1$

The equation of the line is

$$y-e = e(x-1)$$

$$\boxed{y = ex}$$

NAME: ____________________ STUDENT NO: ____________________

4. *[15 marks]*

Use Newton's method to find (to one place after the decimal point) a value of x so that $f'(x) = 0$ where

$$f(x) = e^{\frac{x^4}{4} - x^2 - 5x + 1}$$

(Begin with the value $x_0 = 2$ and go as far as calculating x_2 before you decide on your answer. Note that $f(x)$ itself is never zero.)

$$f'(x) = (x^3 - 2x - 5)e^{\frac{x^4}{4} - x^2 - 5x + 1}$$

The easy way is to note that $e^{\frac{x^4}{4} - x^2 - 5x + 1}$ can never be zero.

So we only need to solve

$x^3 - 2x - 5 = 0$ using Newton's method.

$$x_{n+1} = x_n - \frac{x_n^3 - 2x_n - 5}{3x_n^2 - 2} = \frac{2x_n^3 + 5}{3x_n^2 - 2}$$

$x_0 = 2$, $x_1 = \frac{21}{10} = 2.1$

To 1 place $\boxed{x = 2.1}$

$$x_2 = \frac{2(2.1)^3 + 5}{3 \cdot (2.1)^2 - 2} \approx 2.0946$$

More complicated is to use the full $f(x)$ and $f'(x)$

Since we are solving $f'(x) = 0$ [not $f(x) = 0$]

$$x_{n+1} = x_n - \frac{f'(x_n)}{f''(x_n)}$$, Let $u = \frac{x^4}{4} - x^2 - 5x + 1$

$$f''(x) = (x^3 - 2x - 5)^2 e^u + (3x^2 - 2)e^u$$

So $$x_{n+1} = x_n - \frac{(x_n^3 - 2x_n - 5)\cancel{e^u}}{(x_n^3 - 2x_n - 5)^2 \cancel{e^u} + (3x_n^2 - 2)\cancel{e^u}}$$

$x_0 = 2$ $\quad x_1 = 2 - \frac{(8-4-5)}{(8-4-5)^2 + (12-2)}$

$$= 2 - \frac{-1}{1+10} = 2\tfrac{1}{11} \approx 2.0909$$

$x_2 \approx 2.0946$ so $\boxed{2.09}$ is good

and to 1 decimal place $\boxed{2.1}$ is good enough