Solved

Department of Mathematics
University of Toronto

WEDNESDAY, December 5, 2007 6:10-8:00 PM
MAT 133Y TERM TEST #2

Calculus and Linear Algebra for Commerce
Duration: 1 hour 50 minutes

Aids Allowed: A non-graphing calculator, with empty memory, to be supplied by student.

Instructions: Fill in the information on this page, and make sure your test booklet contains 10 pages. In addition, you should have a **multiple-choice answer sheet**, on which you should fill in your name, number, tutorial time, tutorial room, and tutor's name.

This test consists of 10 multiple choice questions, and 4 written-answer questions.
For the **multiple choice questions** you can do your rough work in the test booklet, but you must record your answer by circling the appropriate letter **on the answer sheet** with your pencil. Each correct answer is worth 4 marks; a question left blank, or an incorrect answer, or two answers for the same question is worth 0. For the **written-answer questions**, present your solutions in the space provided. The value of each written-answer question is indicated beside it.
ENCLOSE YOUR FINAL ANSWER IN A BOX AND WRITE IT IN INK.

TOTAL MARKS: 100

FAMILY NAME: ______________________

GIVEN NAME: ______________________

STUDENT NO: ______________________

SIGNATURE: ______________________

TUTORIAL TIME and ROOM: ______________________

REGCODE and TIMECODE: ______________________

T.A.'S NAME: ______________________

Regcode	Timecode	Room	Regcode	Timecode	Room
T0101A	M9A	SS1084	T0501D	W3D	SS1088
T0101B	M9B	SS1086	T0601A	R4A	LM 157
T0101C	M9C	SS1087	T0601B	R4B	SS1083
T0201A	M3A	SS2108	T0701A	F2A	SS1086
T0201B	M3B	RW 143	T0701B	F2B	SS2106
T0201C	M3C	SS1083	T0701C	F2C	SS2108
T0201D	M3D	RW 142	T0801A	F3A	MP 134
T0301A	T3A	SS1084	T0801B	F3B	MP 118
T0301B	T3B	SS2108	T5101A	M5A	MP 118
T0401A	W9A	SS1084	T5101B	M5B	WI 523
T0401B	W9B	SS1073	T5201A	M6A	LM 162
T0501A	W3A	SS1086	T5201B	M6B	SS2106
T0501B	W3B	SS1083			
T0501C	W3C	SS2106			

FOR MARKER ONLY	
Multiple Choice	
B1	
B2	
B3	
B4	
TOTAL	

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PART A. Multiple Choice

1. *[4 marks]*

$$\lim_{x\to -2} \frac{(x+3)^2-1}{x^2+2x} = \lim_{x\to -2} \frac{x^2+6x+8}{x(x+2)} = \lim_{x\to -2} \frac{\cancel{(x+2)}(x+4)}{\cancel{(x+2)}x}$$

$$= \frac{-2+4}{-2} = -1$$

[L'Hôpital's Rule is not necessary here.]

A. 0

B. 1

C. -1

D. $+\infty$

E. $-\infty$

2. *[4 marks]*

$$\lim_{x\to -\infty} \frac{\sqrt{9x^2-x+6}}{x} = \lim_{x\to -\infty} \frac{\sqrt{x^2(9-\frac{1}{x}+\frac{6}{x^2}}}{x}$$

but $\sqrt{x^2} = -x$ when $x<0$

[L'Hôpital's Rule is not necessary here.]

$$= \lim_{x\to -\infty} -\sqrt{9-\frac{1}{x}+\frac{6}{x^2}}$$

$$= -3$$

A. -1

B. 1

C. -3

D. 3

E. $-\infty$

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3. *[4 marks]*

$$\lim_{x \to +\infty} \frac{4^{x+1} + 3^x}{1 - 4^x} = \lim_{x \to \infty} \frac{4 + \left(\frac{3}{4}\right)^x}{\frac{1}{4^x} - 1}$$

[L'Hôpital's Rule is not necessary here.]

$\left(\frac{3}{4}\right)^x \to 0$ and $\frac{1}{4^x} \to 0$

A. 4

B. -4

C. -1

D. $+\infty$

E. $-\infty$

$= \frac{4}{-1} = -4$

4. *[4 marks]*
If

$$f(x) = \begin{cases} x^2 - 7 & \text{when } x \le 3 \\ x + c & \text{when } x > 3 \end{cases}$$

then $\lim_{x \to 3} f(x)$

A. exists for all values of c

B. exists only when $c > 0$

C. exists only when $c = 0$

D. exists only when $c = -1$

E. does not exist for any value of c

$\lim_{x \to 3^-} f(x) = \lim_{x \to 3^-} x^2 - 7 = 2$

$\lim_{x \to 3^+} f(x) = \lim_{x \to 3^+} x + C = 3 + C$

For $\lim_{x \to 3} f(x)$ to exist, both of these must be equal.

The limit exists only when $3 + C = 2$

i.e. $C = -1$

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5. *[4 marks]*
If $f(x) = x^3 e^{2-2x}$, then $f'(1) =$

A. 1

B. -6

C. -5

D. 6

E. 5

$$f'(x) = 3x^2 e^{2-2x} + x^3 e^{2-2x}(-2)$$

$$f'(1) = 3e^0 - 2e^0 = 1$$

6. *[4 marks]*
If $g(x) = \dfrac{1 + \ln x}{1 + x}$, then $g'(1) =$

A. $\dfrac{3}{4}$

B. $\ln 2$

C. 1

D. $\dfrac{1}{2}$

E. $\dfrac{1}{4}$

$$g'(x) = \frac{(1+x)\frac{1}{x} - (1+\ln x)}{(1+x)^2}$$

$$g'(1) = \frac{2-1}{2^2} = \frac{1}{4}$$

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7. *[4 marks]*
The equation of the tangent line to the curve $y = \ln(\ln x)$ at $(e, 0)$ is

A. $y = ex - e$

B. $y - e = x$

C. $y = x + 1$

D. $y = \dfrac{x}{e} - 1$

E. $x + y = 1$

$y' = \frac{1}{\ln x} \cdot \frac{1}{x}$

$y'(e) = \frac{1}{\ln e} \cdot \frac{1}{e} = \frac{1}{e}$

$y(e) = \ln \ln e = \ln 1 = 0$

$y - 0 = \frac{1}{e}(x - e)$

$y = \frac{x}{e} - 1$

8. *[4 marks]*
If $y = \dfrac{\sqrt{x}\, e^{x^2}}{(x^2+1)^{10}}$, then $y' =$

A. $\dfrac{1}{2}x^{-\frac{1}{2}} e^{x^2}(x^2+1)^{10} + x\sqrt{x}\, e^{x^2}(x^2+1)^{10} - \dfrac{20xe^{x^2}}{(x^2+1)^9}$

B. $\ln(x^2+1) + e^{x^2} + \dfrac{1}{2}\ln x$

C. $10(x^2+1)^9\sqrt{x}\, e^x + \dfrac{1}{2} \cdot \dfrac{\sqrt{x} \cdot 2x\, e^{x^2}}{(x^2+1)^{10}}$

D. $\sqrt{x}\, e^{x^2}(x^2+1)^{10} + \dfrac{1}{2\sqrt{x}} e^{x^2}(x^2+1)^{10} + 2x^{\frac{3}{2}} e^{x^2}$

E. $\dfrac{\sqrt{x}\, e^{x^2}}{(x^2+1)^{10}}\left(\dfrac{1}{2x} + 2x - \dfrac{20x}{x^2+1}\right)$

$\ln y = \frac{1}{2}\ln x + x^2 - 10\ln(x^2+1)$

$\frac{1}{y}y' = \frac{1}{2x} + 2x - \frac{20x}{x^2+1}$

$y' = y\left[\frac{1}{2x} + 2x - \frac{20x}{x^2+1}\right]$. Substituting for y gives E.

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9. *[4 marks]*

If $y = \sqrt{2x+3}$, then $\dfrac{d^3y}{dx^3} = y''' =$

A. $\dfrac{3}{2\sqrt{2x+3}}$

B. 0

C. $-(2x+3)^{-\frac{5}{2}}$

D. $-\dfrac{1}{2(2x+3)\sqrt{2x+3}}$

E. $3(2x+3)^{-\frac{5}{2}}$

$y = (2x+3)^{\frac{1}{2}}$

$y' = \frac{1}{2}(2x+3)^{-\frac{1}{2}} \cdot 2$

$= (2x+3)^{-\frac{1}{2}}$

$y'' = -\frac{1}{2}(2x+3)^{-\frac{3}{2}} \cdot 2$

$= -(2x+3)^{-\frac{3}{2}}$

$y''' = \frac{3}{2}(2x+3)^{-5/2} \cdot 2$

$= 3(2x+3)^{-\frac{5}{2}}$

10. *[4 marks]*

If $\sqrt{x+y} = 1 + x^2y^2$ defines y implicitly as a function of x, then $y' =$

A. $\dfrac{xy^2\sqrt{x+y} - 1}{1 - x^2y\sqrt{x+y}}$

B. $\dfrac{4xy^2\sqrt{x+y} - 1}{1 - 4x^2y\sqrt{x+y}}$

C. $\dfrac{2xy^2}{2x^2y - \dfrac{1}{2\sqrt{x+y}}}$

D. $\dfrac{4xy}{\sqrt{x+y}}$

E. $4xy^2\sqrt{x+y} - 1$

$\frac{1}{2\sqrt{x+y}}(1+y') = 2xy^2 + 2x^2yy'$

$y'\left[\frac{1}{2\sqrt{x+y}} - 2x^2y\right] = 2xy^2 - \frac{1}{2\sqrt{x+y}}$

$y' = \dfrac{2xy^2 - \frac{1}{2\sqrt{x+y}}}{\frac{1}{2\sqrt{x+y}} - 2x^2y}$

$y' = \dfrac{4xy^2\sqrt{x+y} - 1}{1 - 4x^2y\sqrt{x+y}}$

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PART B. Written-Answer Questions

1. *[15 marks]*

Let

$$f(x) = \begin{cases} 2^{1+x} - k & \text{if } x < -1 \\ \dfrac{x^2-1}{(x-1)(x-2)} & \text{if } -1 \le x < 1 \\ 2x - 4 & \text{if } x > 1. \end{cases}$$

[5] (a) Is f continuous at $x = 2$? (Justify your answer.) Yes.

$\lim_{x \to 2} f(x) = \lim_{x \to 2} (2x-4) = 0$ since $2x-4$ is nice.

and $f(2) = 0$.

i.e. $\boxed{\lim_{x \to 2} f(x) = f(2)}$.

[5] (b) Why is f not continuous at $x = 1$? How would you define $f(1)$ to make f continuous at $x = 1$?

f is not cont. at $x=1$ because it is not defined at $x=1$. However,

$\lim_{x \to 1^-} f(x) = \lim_{x \to 1^-} \frac{x^2-1}{(x-1)(x-2)} = \lim_{x \to 1^-} \frac{x+1}{x-2} = -2$

and $\lim_{x \to 1^+} f(x) = \lim_{x \to 1^+} 2x-4 = -2$ as well.

Just define $\boxed{f(1) = -2}$, and f is cont. at $x=1$.

[5] (c) Find k so that f is continuous at $x = -1$.

$f(-1) = \lim_{x \to -1^+} f(x) = 0$ by inspection

$\lim_{x \to -1^-} f(x) = \lim_{x \to -1^-} 2^{1+x} - k = 2^0 - k = 1 - k$

f will be cont at $x = -1$ if $\boxed{k = 1}$

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2. *[16 marks]*

Solve the following inequalities for x

[8] (a) $(x^2 - 4)\ln x < 0$

$(x^2-4)\ln x$ is defined only for $x>0$, and there it is cont. and $=0$ at $x=2$ and $x=1$ only.

Interval	point	
$(0,1)$	$\frac{1}{2}$ e.g.	+
$(1,2)$	$\frac{3}{2}$ e.g.	−
$(2,\infty)$	10^{53}	+

$\boxed{1<x<2}$ or $\boxed{x \in (1,2)}$

[8] (b) $\dfrac{|x+2|-2}{x-1} \geq 0$

$|x+2|-2=0$ when $|x+2|=2$ only.

$x+2=2 \Rightarrow x=0$

$x+2=-2 \Rightarrow x=-4$

and the only discont. is at $x=1$

Interval	point	
$(-\infty,-4)$	-1000	−
$(-4,0)$	-2	+
$(0,1)$	$\frac{1}{2}$	−
$(1,\infty)$	1000	+

$\dfrac{998-2}{-1001}$

$(5\frac{1}{2}-2)/-\frac{1}{2}$

>0 on $(-4,0)$ and $(1,\infty)$

$=0$ at -4 and 0

so ≥ 0 on $\boxed{[-4,0] \text{ and } (1,\infty)}$

or $\boxed{[-4,0] \cup (1,\infty)}$

or $\boxed{-4 \leq x \leq 0 \text{ or } x>1}$

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3. *[15 marks]*

The demand function for a product is

$$p = \frac{12}{\ln(q+3)}$$

Find the **marginal revenue** and the **point elasticity of demand** when $q = 5$.
[Please remember that this is a calculus course.]

$$\frac{dp}{dq} = \frac{-12}{[\ln(q+3)]^2(q+3)} = -\frac{12}{8(\ln 8)^2} = -\frac{3}{2(\ln 8)^2} \text{ at } q=5$$

$$p = \frac{12}{\ln 8} \text{ at } q=5.$$

$$R = pq \quad \frac{dR}{dq} = p + q\frac{dp}{dq} = \frac{12}{\ln 8} - \frac{15}{2(\ln 8)^2} \text{ at } q=5$$

$$\boxed{\frac{dR}{dq} \approx 4.036} \text{ at } q=5$$

$$\eta = \frac{p}{q\frac{dp}{dq}} = \frac{12}{5\ln 8\left(\frac{-12}{8(\ln 8)^2}\right)} = -\frac{8\ln 8}{5} \text{ at } q=5$$

$$\boxed{\eta \approx -3.327} \text{ at } q=5$$

Alternatively: If you compute either $\frac{dR}{dq}$ or η only, you can use $\frac{dR}{dq} = p\left(1+\frac{1}{\eta}\right)$ to get the other.

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4. *[14 marks]*

Use Newton's Method to approximate the root of the equation $x^4 + x - 4 = 0$ in the interval $(1, 2)$ to six decimal places.

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} \quad \text{if } f(x) = x^4 + x - 4$$

$$= x_n - \frac{x_n^4 + x_n - 4}{4x_n^3 + 1}$$

$$x_{n+1} = \frac{3x_n^4 + 4}{4x_n^3 + 1} \quad \text{and} \quad \begin{array}{l} f(1) < 0 \\ f(2) > 0. \end{array}$$

If we start at $x_0 = 1.5$

$x_1 = 1.323275862$

$x_2 = 1.285346065$

$x_3 = 1.283784219$

$x_4 = 1.283781666$

$\boxed{x_5 = 1.283781\boxed{666}}$ to 8 or 9 decimal places actually.

also OK is $x_5 = 1.283782$ to 6 places

If we start at $x_0 = 1$

$x_1 = \frac{7}{5} = 1.4$

$x_2 = 1.296325985$

$x_3 = 1.283943953$

$x_4 = 1.283781693$

$x_5 = \boxed{1.283781}666$

which is the same as the previous x_5 to 9 decimal places. Note that if you really wanted 9 decimal places, starting at $x_0 = 1$ would require the computation of x_6 to be sure.