Solved

Department of Mathematics
University of Toronto

WEDNESDAY, December 6, 2006 6:10-8:00 PM
MAT 133Y TERM TEST #2

Calculus and Linear Algebra for Commerce
Duration: 1 hour 50 minutes

Aids Allowed: A non-graphing calculator, with empty memory, to be supplied by student.

Instructions: Fill in the information on this page, and make sure your test booklet contains 11 pages. In addition, you should have a **multiple-choice answer sheet**, on which you should fill in your name, number, tutorial time, tutorial room, and tutor's name.

This test consists of 10 multiple choice questions, and 4 written-answer questions.
For the **multiple choice questions** you can do your rough work in the test booklet, but you must record your answer by circling the appropriate letter **on the answer sheet** with your pencil. Each correct answer is worth 4 marks; a question left blank, or an incorrect answer, or two answers for the same question is worth 0. For the **written-answer questions**, present your solutions in the space provided. The value of each written-answer question is indicated beside it. ENCLOSE YOUR FINAL ANSWER IN A BOX AND WRITE IT IN INK.

TOTAL MARKS: 100

FAMILY NAME: ______________________

GIVEN NAME: ______________________

STUDENT NO: ______________________

SIGNATURE: ______________________

TUTORIAL TIME and ROOM: ______________________

REGCODE and TIMECODE: ______________________

T.A.'S NAME: ______________________

Regcode	Timecode	Room	**Regcode**	Timecode	Room
T0101A	M9A	LM 155	T0501D	W3D	BF 323
T0101B	M9B	SS1074	T0601A	R4A	MP 137
T0101C	M9C	SS1084	T0601B	R4B	RW 143
T0201A	M3A	MS3163	T0701A	F2A	MP 137
T0201B	M3B	SS1087	T0701B	F2B	SS1087
T0201C	M3C	MP 134	T0701C	F2C	LM 162
T0201D	M3D	MP 137	T0801A	F3A	SS1083
T0301A	T3A	SS2110	T0801B	F3B	SS1073
T0301B	T3B	SS1083	T5101A	M5A	SS1069
T0401A	W9A	SS1084	T5101B	M5B	SS2108
T0401B	W9B	SS2127	T5201A	M6A	SS2110
T0501A	W3A	SS2102			
T0501B	W3B	RW 110			
T0501C	W3C	MP 203			

FOR MARKER ONLY	
Multiple Choice	
B1	
B2	
B3	
B4	
TOTAL	

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PART A. Multiple Choice

1. *[4 marks]*

$$\lim_{x\to 1}\left(\frac{4x}{x^2-1}-\frac{2}{x-1}\right) =$$

$\lim\limits_{x\to 1} \frac{4x-2(x+1)}{x^2-1} = \lim\limits_{x\to 1} \frac{2(x-1)}{x^2-1}$

$= \lim\limits_{x\to 1} \frac{2}{x+1} = \frac{2}{1+1} = 1$ B

A. equals -2

B. equals 1

C. equals 2

D. equals 0

E. does not exist

2. *[4 marks]*

$$\lim_{x\to -\infty}\frac{2x^2-6x}{x^2-4x+3} =$$

$\lim\limits_{x\to -\infty} \frac{\cancel{x^2}(2-\frac{6}{x})}{\cancel{x^2}(1-\frac{4}{x}+\frac{3}{x^2})} = 2$ D

A. $+\infty$

B. 1

C. -2

D. 2

E. $-\infty$

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3. *[4 marks]*
If \$500 grows to \$1,000 in 8 years then interest must be compounded continuously at an annual rate that is closest to

A. 7.72%

B. 6.25%

C. 8.66%

D. 9.13%

E. 9.05%

$P(t) = P(0)e^{rt}$

$1000 = 500e^{8r}$

$\ln 2 = 8r$

$r = \frac{1}{8}\ln 2 \approx .0866$

8.66% C

4. *[4 marks]*
The function given by $f(x) = |x - 2|$ on the interval $0 \le x < 4$ has

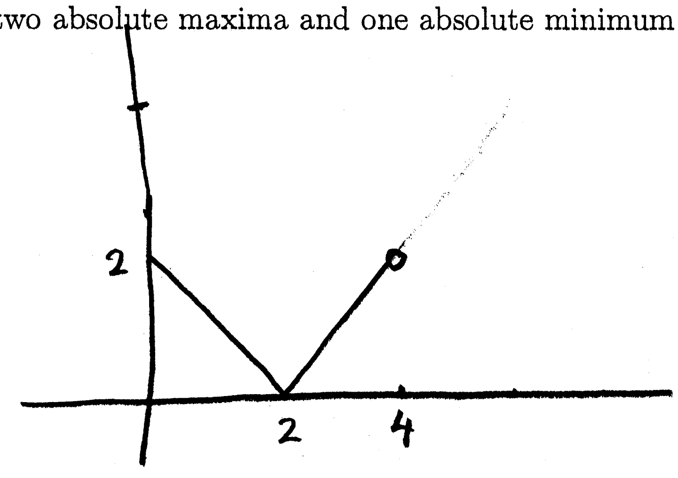
A. no absolute extrema

B. one absolute minimum but no absolute maximum

C. one absolute maximum but no absolute minimum

D. one absolute maximum and one absolute minimum

E. two absolute maxima and one absolute minimum



max at $x = 0$ only
min at $x = 2$ only

D

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5. *[4 marks]*

$$f(x) = \begin{cases} \dfrac{\ln(2+x) - \ln 2}{x} & \text{when} \quad x > 0 \\ a & \text{when} \quad x = 0 \\ e^{x-b} & \text{when} \quad x < 0 \end{cases}$$

If f is continuous at $x = 0$ then

A. $a = \ln 2$ and $b = 0$

B. $a = 1$ and $b = 1$

C. $a = \dfrac{1}{2}$ and $b = 2$

D. $a = 2$ and $b = \frac{1}{2}$

E. $a = \dfrac{1}{2}$ and $b = \ln 2$

$\lim_{x \to 0^+} f(x) = \frac{d}{dt}\Big|_{t=2} \ln x = \frac{1}{2}$

$\qquad\qquad \shortparallel$

by cont $f(0) = a$

$\qquad\qquad \shortparallel$

$\lim_{x \to 0^-} f(x) = e^{-b}$

so $\boxed{a = \frac{1}{2}} = e^{-b}$ $\quad e^b = 2$ $\quad \boxed{b = \ln 2}$

E

6. *[4 marks]*

If $y = x\sqrt{4x+3}$, then the tangent line to the curve $y = f(x)$ is horizontal

A. never

B. at $x = 0$, $y = 0$

C. at $x = -\dfrac{3}{4}$, $y = 0$

D. at $x = 1$, $y = \sqrt{7}$

E. at $x = -\dfrac{1}{2}$, $y = -\dfrac{1}{2}$

$y' = \sqrt{4x+3} + \dfrac{x \cdot 4}{2\sqrt{4x+3}}$

$y' = \dfrac{6x+3}{\sqrt{4x+3}}$

$y' = 0$ when $x = -\frac{1}{2}$

then $y = -\frac{1}{2}\sqrt{-2+1} = -\frac{1}{2}$

E

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7. *[4 marks]*
Find all λ, for which $y = e^{\lambda x}$ satisfies the equation

$$y'' + 6y' + 8y = 0$$

A. $\lambda = -2$ only

B. $\lambda = 1$ and $\lambda = 4$

C. $\lambda = -2$ and $\lambda = -4$

D. $\lambda = 0$ only

E. There is no such λ

$y' = \lambda e^{\lambda x}$

$y'' = \lambda^2 e^{\lambda x}$

$y'' + 6y' + 8y$

$= (\lambda^2 + 6\lambda + 8) e^{\lambda x} = 0$

$e^{\lambda x} \neq 0$ so $\lambda^2 + 6\lambda + 8 = 0$

$(\lambda + 2)(\lambda + 4) = 0$

$\lambda = -2$ and $\lambda = -4$

C

8. *[4 marks]*
If $h(x) = g\big(f(x)\big)$ and $u = f(x)$, then $h''(x) =$

A. $g''(u) \cdot f'(x) + g'(u) \cdot \big(f''(x)\big)^2$

B. $g''(u) \cdot f'(x) + g'(u) \cdot f''(x)$

C. $g''(u) \cdot f'(x)$

D. $g''(u) \cdot \big(f'(x)\big)^2 + g'(u) \cdot f''(x)$

E. $g''(u) \cdot f''(x)$

$h'(x) = g'(f(x)) f'(x)$

$h''(x) = g''(f(x)) f'(x) f'(x) + g'(f(x)) f''(x)$

so D

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9. *[4 marks]*

The derivative of

$$f(x) = (2x^2 - 3x + 5)^{\sqrt{x^2-1}+e^x}$$

is equal to

A. $(2x^2 - 3x + 5)^{\sqrt{x^2-1}+e^x} \cdot \ln(2x^2 - 3x + 5)$

B. $(2x^2-3x+5)^{\sqrt{x^2-1}+e^x}\left(\left(\frac{x}{\sqrt{x^2-1}} + e^x\right)\ln(2x^2 - 3x + 5) + (\sqrt{x^2-1} + e^x)\frac{4x-3}{2x^2-3x+5}\right)$

C. $\left(\frac{x}{\sqrt{x^2-1}} + e^x\right)\ln(2x^2 - 3x + 5)(2x^2 - 3x + 5)^{\sqrt{x^2-1}+e^x}$

D. $(2x^2 - 3x + 5)^{\sqrt{x^2-1}+e^x}\left(\sqrt{x^2-1} + e^x\right)\left(\frac{x}{\sqrt{x^2-1}} + e^x\right)$

E. $\frac{f'(x)}{f(x)}\ln[f(x)]$

$\ln f = (\sqrt{x^2-1} + e^x)\ln(2x^2-3x+5)$

$\frac{1}{f}f' = \left(\frac{x}{\sqrt{x^2-1}} + e^x\right)\ln(2x^2-3x+5) + \frac{(\sqrt{x^2-1}+e^x)(4x-3)}{2x^2-3x+5}$

To get f', multiply by f, so B

10. *[4 marks]*

If $y(x)$ satisfies $y^x = x^y$, what is the value of $y'(x)$ when $(x, y) = (4, 2)$?

A. $\frac{2 - \ln 4}{\frac{1}{2} - \ln 2}$

B. $\frac{\frac{1}{2} - \ln 2}{2 - \ln 4}$

C. $\frac{2 + \ln 2}{\frac{1}{2} + \ln 4}$

D. $\frac{2 - \ln 2}{\frac{1}{2} - \ln 4}$

E. $\frac{\frac{1}{2} - \ln 4}{2 - \ln 2}$

$x\ln y = y\ln x$

$\ln y + \frac{x}{y}y' = y'\ln x + \frac{y}{x}$

At $x=4$, $y=2$

$\ln 2 + 2y' = y'\ln 4 + \frac{1}{2}$

$y'(2-\ln 4) = \frac{1}{2} - \ln 2$

$y' = \frac{\frac{1}{2}-\ln 2}{2-\ln 4}$ B

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PART B. Written-Answer Questions

1. *[14 marks]*

[7] (a) Solve the following inequality for x

$$\text{Let } f(x) = \frac{\ln x}{3 - e^x} \leq 0$$

f is defined only for $x > 0$ and $x \neq \ln 3$

$f(x) = 0$ at $x = 1$. So f cont. and $\neq 0$

on $(0, 1)$, $(1, \ln 3)$, $(\ln 3, \infty)$

On $(0, 1]$: $\ln x \leqslant 0$ and $e^x < 3$ so $\frac{\ln x}{3 - e^x} \leqslant 0$

On $(\ln 3, \infty)$: $\ln x > 0$ and $e^x > 3$ so $\frac{\ln x}{3 - e^x} < 0$

On $(1, \ln 3)$ $\ln x > 0$ and $e^x < 3$

so $\frac{\ln x}{3 - e^x} > 0$

So $\boxed{x \text{ in } (0, 1] \cup (\ln 3, \infty)}$

or: $\boxed{0 < x \leq 1 \text{ or } \ln 3 < x}$

[7] (b) If $r = 300q^2 - q^3$ is the total revenue when q units are sold then find when marginal revenue is positive.

$$MR = \frac{dr}{dq} = 600q - 3q^2$$

$$= 3q(200 - q)$$

$MR > 0$ when $\boxed{q < 200}$

and bigger than zero of course.

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2. *[18 marks]*

In order to sell q metres of photographic film per day (where $q > 0$) the manufacturer must set its price at $p(q) = q^{\frac{1}{4}} e^{-\frac{q}{12}}$ dollars per metre sold.

[4] (a) Find the manufacturer's marginal revenue as a function of q.

$$\frac{dr}{dq} = \frac{d}{dq}(pq) = \frac{d}{dq}\left(q^{\frac{5}{4}} e^{-\frac{q}{12}}\right)$$

$$= \boxed{\frac{5}{4} q^{\frac{1}{4}} e^{-\frac{q}{12}} - \frac{1}{12} q^{\frac{5}{4}} e^{-\frac{q}{12}}}$$

or $$= \boxed{\left(\frac{5}{4} - \frac{q}{12}\right) q^{\frac{1}{4}} e^{-\frac{q}{12}}}$$

$$= \left(\frac{5}{4} - \frac{q}{12}\right) p(q) \text{ notice}$$

[5] (b) Find the relative rate of change of revenue as a function of q.

$$\frac{1}{r}\frac{dr}{dq} = \frac{1}{r}\left(\frac{5}{4} - \frac{q}{12}\right) p(q)$$

$$= \frac{1}{\cancel{r}}\left(\frac{5}{4} - \frac{q}{12}\right) \frac{\cancel{pq}}{q}$$

$$= \boxed{\frac{5}{4q} - \frac{1}{12}}$$

or $$= \boxed{\frac{15 - q}{12q}}$$

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2.

[6] (c) Find the elasticity of demand as a function of q.

$$\eta \approx \frac{\frac{\Delta q}{q}}{\frac{\Delta p}{p}} \quad \text{so} \quad \eta = \frac{\frac{p}{q}}{\frac{dp}{dq}}$$

$$\frac{p}{q} = q^{-\frac{3}{4}} e^{-\frac{q}{12}} \qquad \frac{dp}{dq} = \frac{1}{4} q^{-\frac{3}{4}} e^{-\frac{q}{12}} - \frac{1}{12} q^{\frac{1}{4}} e^{-\frac{q}{12}}$$

$$= \frac{e^{-\frac{q}{12}}(3-q)}{12 q^{\frac{3}{4}}}$$

$$\eta = \frac{\cancel{q^{-\frac{3}{4}}} \cancel{e^{-\frac{q}{12}}}}{\frac{\cancel{q^{-\frac{3}{4}}} \cancel{e^{-\frac{q}{12}}} (3-q)}{12}} = \boxed{\frac{12}{3-q}}$$

[3] (d) For which q is demand inelastic?

Inelastic when $|\eta| < 1$, i.e. $-1 < \eta < 1$

$$1 > \frac{12}{3-q} > -1$$

$$\frac{12}{3-q} + 1 > 0 \qquad \text{i.e.} \quad \frac{15-q}{3-q} > 0$$

so $q > 15$ will do. So will $q < 3$.

But if $q < 3$, $3 - q > 3$ so $\frac{12}{3-q} > 4$ no good.

Hence $\boxed{q > 15}$

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3. *[15 marks]*

[7] (a) Given that $x^4 + y^4 = 4y$, find $\dfrac{dy}{dx}$ in terms of x and y only. There is no need to "simplify".

$$4x^3 + 4y^3 y' = 4y'$$

$$(4y^3 - 4)y' = -4x^3 \qquad (y^3 - 1)y' = x^3$$

$$\boxed{y' = \frac{x^3}{1-y^3}}$$

[8] (b) Given that $x^4 + y^4 = 4y$, find $\dfrac{d^2y}{dx^2}$ in terms of x and y only. There is no need to "simplify".

Starting from the answer

$$y'' = \frac{(1-y^3)3x^2 - x^3(-3y^2 y')}{(1-y^3)^2}$$

$$y'' = \boxed{\frac{(1-y^3)3x^2 + 3x^3 y^2 \dfrac{x^3}{1-y^3}}{(1-y^3)^2}}$$

or $$\frac{(1-y^3)^2 \cdot 3x^2 + 3x^6 y^2}{(1-y^3)^3}$$

Starting from one line higher than the answer

$$(3y^2 y')y' + (y^3 - 1)y'' = 3x^2$$

$$y'' = \frac{3x^2 - 3y^2(y')^2}{y^3 - 1}$$

$$= \frac{3x^2 - 3y^2\left(\dfrac{x^3}{1-y^3}\right)^2}{y^3 - 1}$$

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4. *[13 marks]*

Find a root of

$$x^5 + 4x + 1 = 0$$

up to 6-decimal-place-accuracy using Newton method and $x_1 = 0$ as the initial estimate.

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

$$= x_n - \frac{x_n^5 + 4x_n + 1}{5x_n^4 + 4}$$

$$x_{n+1} = \frac{4x_n^5 - 1}{5x_n^4 + 4}$$

$$x_2 = -\frac{1}{4} = -.25$$

$$x_3 = -.249757045$$

$$x_4 = -.249757043$$

The answer has stabilized out to 7 (even 8) places

$$\boxed{x_4 = -.249757}$$ to 6 places is good enough

If $f(x) = x^5 + 4x + 1$

$f(x_4) \approx 1.7 \times 10^{-7}$ pretty good!