Solved

Department of Mathematics
University of Toronto

THURSDAY, DECEMBER 15, 2005 9:00-11:00 AM
MAT 133Y TERM TEST #2

Calculus and Linear Algebra for Commerce
Duration: 2 hours

Aids Allowed: A non-graphing calculator, with empty memory, to be supplied by student.

Instructions: Fill in the information on this page, and make sure your test booklet contains 10 pages. In addition, you should have a **multiple-choice answer sheet**, on which you should fill in your name, number, tutorial time, tutorial room, and tutor's name.

This test consists of 10 multiple choice questions, and 4 written-answer questions.
For the **multiple choice questions** you can do your rough work in the test booklet, but you must record your answer by circling the appropriate letter **on the answer sheet** with your pencil. Each correct answer is worth 4 marks; a question left blank, or an incorrect answer, or two answers for the same question is worth 0. For the **written-answer questions**, present your solutions in the space provided. The value of each written-answer question is indicated beside it. ENCLOSE YOUR FINAL ANSWER IN A BOX AND WRITE IT IN INK.

TOTAL MARKS: 100

FAMILY NAME: ______________________

GIVEN NAME: ______________________

STUDENT NO: ______________________

SIGNATURE: ______________________

TUTORIAL TIME and ROOM: ______________________

REGCODE and TIMECODE: ______________________

T.A.'S NAME: ______________________

Regcode	Timecode	Room	**Regcode**	Timecode	Room
T0101A	M9A	LM 123	T0501D	W3D	UCA101
T0101B	M9B	LM 157	T0601A	R4A	LM 123
T0101C	M9C	WI 523	T0601B	R4B	LM 157
T0201A	M3A	WO 35	T0701A	F2A	LM 157
T0201B	M3B	SS2128	T0701B	F2B	MP 118
T0201C	M3C	WI 524	T0701C	F2C	SS1084
T0201D	M3D	UC 52	T0801A	F3A	MP 118
T0301A	T3A	UC 87	T0801B	F3B	WI 523
T0301B	T3B	UC 256	T5101A	M5A	LM 155
T0401A	W9A	LM 123	T5101B	M5B	WI 523
T0401B	W9B	LM 157	T5201A	M6A	LM 123
T0501A	W3A	UC 244			
T0501B	W3B	UC 328			
T0501C	W3C	UC 52			

FOR MARKER ONLY	
Multiple Choice	
B1	
B2	
B3	
B4	
TOTAL	

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PART A. Multiple Choice

1. *[4 marks]*

   $\lim_{x\to 2} \dfrac{x^2-3x+2}{\sqrt{x+2}-2}$ is

   A. 0

   B. 4

   C. -4

   D. ∞

   E. $-\infty$

$$\lim_{x\to 2} \frac{\cancel{(x-2)}(x-1)(\sqrt{x+2}+2)}{\cancel{(x+2-4)}}$$

$$= \lim_{x\to 2} (x-1)(\sqrt{x+2}+2)$$

$$= \boxed{4}$$

2. *[4 marks]*

   If $\dfrac{x^2(x+1)}{x-2} \le 0$ then x is in

   A. $(-\infty,\ -1] \cup (2,\ \infty)$

   B. $(-\infty,\ -1] \cup [0,\ 2)$

   C. $[-1,\ 0] \cup (2,\ \infty)$

   D. $(-1,\ 0) \cup (0,\ 2)$

   E. $[-1,\ 2)$

Int	f
$(-\infty, -1)$	+
$(-1, 0)$	−
$(0, 2)$	−
$(2, \infty)$	+

$f < 0$ on $(-1,0) \cup (0,2)$

$f = 0$ at $x = -1$ and $x = 0$

So $f \le 0$ on $\boxed{[-1, 2)}$

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3. *[4 marks]*
If \$10,000 is deposited in a savings account that earns interest at an annual rate of 5% compounded continuously, the value of the account after four years will be closest to:

A. \$12,000
B. \$12,214
C. \$12,137
D. \$11,846
E. \$13,515

$P = 10{,}000\, e^{.05 \times 4}$

$\approx \boxed{\$\,12{,}214}$

4. *[4 marks]*
If $f(x) = \dfrac{x}{\sqrt{1-x^2}}$, then $f'\left(\dfrac{3}{5}\right) =$

A. $\dfrac{125}{64}$
B. $\dfrac{16}{25}$
C. $\dfrac{35}{64}$
D. $\dfrac{25}{16}$
E. $\dfrac{64}{125}$

$$f'(x) = \frac{\sqrt{1-x^2} - \dfrac{x(-\cancel{2}x)}{\cancel{2}\sqrt{1-x^2}}}{1-x^2}$$

$$= \frac{1}{(1-x^2)^{3/2}}$$

$$f'\left(\tfrac{3}{5}\right) = \frac{1}{\left(1-\frac{9}{25}\right)^{3/2}} = \left(\frac{25}{16}\right)^{3/2}$$

$$= \left(\frac{5}{4}\right)^3$$

$$= \boxed{\frac{125}{64}}$$

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5. *[4 marks]*
If the function $f(x)$ is differentiable at all real x, then $\lim_{x\to 2} \dfrac{f(\ln x) - f(\ln 2)}{x-2} =$

A. $\frac{1}{2}\ln(f'(2))$

B. $f'(\ln 2)$

C. $\frac{1}{2}\ln(f(2))$

D. $\ln(f'(2))$

E. $\frac{1}{2}f'(\ln 2)$

Let $g(x) = f(\ln x)$

By definition

$$g'(2) = \lim_{x\to 2} \frac{g(x)-g(2)}{x-2} = \lim_{x\to 2} \frac{f(\ln x) - f(\ln 2)}{x-2}$$

But by the chain-rule $g'(x) = f'(\ln x)\cdot \frac{1}{x}$

So $g'(2) = \boxed{\dfrac{f'(\ln 2)}{2}}$

6. *[4 marks]*
Let u and v denote functions whose values and derivatives at $x = 1, 2$, and 3 are given below.

$u(1) = 2,\quad u(2) = 3,\quad u(3) = 1 \qquad v(1) = 3,\quad v(2) = 1,\quad v(3) = 2$

$u'(1) = 4,\quad u'(2) = 8,\quad u'(3) = 16 \qquad v'(1) = 16,\quad v'(2) = 8,\quad v'(3) = 2$

If $w(x) = v(u(x))$, then $w'(2) =$

A. 8

B. 16

C. 32

D. 64

E. 128

$$w'(x) = v'(u(x))\,u'(x)$$
$$w'(2) = v'(u(2))\,u'(2) = v'(3)\cdot 8 = 2\cdot 8 = \boxed{16}$$

7. *[4 marks]*
Which one of the following is true for the demand equation given by $pq = 200$?

A. It has unit elasticity for all $p > 0,\ q > 0$.

B. It is elastic for all $p > 0,\ q > 0$.

C. It is inelastic for all $p > 0,\ q > 0$.

D. It is elastic for **only** some of the values p and q where $p > 0,\ q > 0$.

E. It is inelastic for **only** some of the values p and q where $p > 0,\ q > 0$.

$$q = \frac{200}{p} \qquad \frac{dq}{dp} = -\frac{200}{p^2}$$

$$\mu = \frac{p}{q}\frac{dq}{dp} = \frac{p}{\frac{200}{p}} \cdot \frac{(-200)}{p^2} = -\frac{p^2}{p^2} = -1$$

8. *[4 marks]*
If Newton's Method is used to estimate a root of the equation

$$x^4 - 2x - 3 = 0$$

and the first approximation, x_1, is taken to be 1, then x_3 is closest to

A. 2.46182

B. 2.58138

C. 3

D. 2.32075

E. 1.87559

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

$$x_{n+1} = x_n - \frac{x_n^4 - 2x_n - 3}{4x_n^3 - 2}$$

$$x_1 = 1 \qquad x_2 = 1 - \frac{-4}{2} = 3$$

$$x_3 = 3 - \frac{81 - 6 - 3}{4 \cdot 27 - 2} \approx \boxed{2,32075}$$

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9. *[4 marks]*
If $f(x) = \ln x$, then $f^{(30)}(x) =$

A. $\dfrac{-(28)(27)(26)\cdots(3)(2)}{x^{29}}$

B. $\dfrac{(28)(27)(26)\cdots(3)(2)}{x^{29}}$

C. $\dfrac{(29)(28)(27)\cdots(3)(2)}{x^{30}}$

D. $\dfrac{-(29)(28)(27)\cdots(3)(2)}{x^{30}}$

E. $\dfrac{-(30)(29)(28)\cdots(3)(2)}{x^{30}}$

$f' = \frac{1}{x}$ $\quad f'' = -\frac{1}{x^2}$

$f^{(3)} = \frac{2}{x^3}$ $\quad f^{(4)} = -\frac{3 \cdot 2}{x^4}$

$f^{(5)} = \frac{4 \cdot 3 \cdot 2}{x^5}$

$f^{(k)} = \frac{(k-1)(k-2)\cdots 2}{x^k} \cdot (-1)^{k-1}$

$f^{(30)}(x) = \frac{29 \cdot 28 \cdots 2}{x^{30}} (-1)^{29}$

but $(-1)^{29} = -1$

10. *[4 marks]*
If $f(x) = x^{2x+3}$, then $f'(2) =$

A. $7 \cdot 2^6$

B. $7 \cdot 2^7$

C. $2^7 \ln 2$

D. $2\ln 2 + \frac{7}{2}$

E. $2^8(\ln 2 + \frac{7}{4})$

$\ln f = (2x+3)\ln x$

$\frac{1}{f(x)} f'(x) = 2\ln x + \frac{2x+3}{x}$

$\frac{1}{f(2)} f'(2) = 2\ln 2 + \frac{7}{2}$

But $f(2) = 2^7$

So $f'(2) = 2^7\left(2\ln 2 + \frac{7}{2}\right)$

$= \boxed{2^8\left(\ln 2 + \frac{7}{4}\right)}$

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PART B. Written-Answer Questions

1. *[15 marks]*
Let

$$f(x) = \begin{cases} ax + \dfrac{1}{x}, & x \le -1 \\ x^2 + \sqrt{x+1}, & -1 < x < 3 \\ b, & x = 3 \\ \dfrac{2x^2 - x - 4}{c + 3x^2}, & x > 3 \end{cases}$$

(a) *[8 marks]*
Find real numbers a, b, c such that f is continuous everywhere.

At $x=-1$, $\lim_{x\to -1^+} f(x) = (-1)^2 + \sqrt{-1+1} = 1$

$\lim_{x\to -1^-} f(x) = -a - 1 = f(-1)$

Cont. if $-a-1 = 1$ $\boxed{a = -2}$

At $x=3$, $\lim_{x\to 3^-} f(x) = 3^2 + \sqrt{3+1} = 11$, $f(3) = b$, so $\boxed{b = 11}$

$\lim_{x\to 3^+} f(x) = \dfrac{2\cdot 9 - 3 - 4}{c+27} = \dfrac{11}{c+27} = f(3) = 11$, so $\boxed{c = -26}$

Note: $x > 3 \Rightarrow -26 + 3x^2 > 0$, so this value of c creates no problems for $f(x)$.

(b) *[7 marks]*
For what real values of a, b and c do $\lim_{x\to\infty} f(x)$ and $\lim_{x\to -\infty} f(x)$ exist, and what are these limits?

$\lim_{x\to-\infty} f(x) = \lim_{x\to-\infty} \left(ax + \frac{1}{x}\right)$ exists only if $\boxed{a=0}$

and then $\lim_{x\to-\infty} f(x) = \lim_{x\to-\infty} \frac{1}{x} = \boxed{0}$

$\lim_{x\to\infty} f(x) = \lim_{x\to\infty} \dfrac{2x^2 - x - 4}{c + 3x^2} = \lim_{x\to\infty} \dfrac{2 - \frac{1}{x} - \frac{4}{x^2}}{\frac{c}{x^2} + 3}$

$= \boxed{\frac{2}{3}}$ $\boxed{\text{no matter what } b \text{ and } c \text{ are}}$

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2. *[17 marks]*
The Acme Nail Manufacturing Company finds it can sell q kilograms of nails per week provided it sets its selling price at $q^{-\frac{1}{3}}$ dollars per kilogram.

[4] (a) Find Acme's marginal revenue.

$$r = pq = q^{-\frac{1}{3}} q = q^{\frac{2}{3}}$$

$$\frac{dr}{dq} = \frac{2}{3} q^{-\frac{1}{3}}$$

[3] (b) Find the relative rate of change of Acme's revenue with respect to q.

$$\frac{1}{r}\frac{dr}{dq} = q^{-\frac{2}{3}} \frac{2}{3} q^{-\frac{1}{3}} = \boxed{\frac{2}{3} q^{-1} = \frac{2}{3q}}$$

[5] (c) If Acme had m employees, they could produce $q(m)$ kilograms of nails per week. Currently, $m = 64$, $q = 125$, and marginal revenue product is 0.4 dollars per week per employee. Find the current value of $\frac{dq}{dm}$. (Recall that marginal revenue product is $\frac{dr}{dm}$, where r denotes revenue.)

$$\frac{dr}{dm} = \frac{dr}{dq}\frac{dq}{dm} \qquad \frac{dr}{dq} = \frac{2}{3}(125)^{-\frac{1}{3}} = \frac{2}{15} \qquad \frac{dr}{dm} = .4$$

$$.4 = \frac{2}{15}\frac{dq}{dm} \Rightarrow \boxed{\frac{dq}{dm} = 3}$$

[5] (d) Find Acme's approximate total weekly output of nails if they decide to increase their workforce from 64 to 70.

$\frac{\Delta q}{\Delta m} \approx 3$ when $m = 64$

$\Delta q \approx 3\Delta m = 3 \times 6$ if m goes from 64 to 70.

$\Delta q = 18$ the change in q

The new total weekly amount is

$125 + 18 = \boxed{143}$

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3. *[15 marks]*
Suppose that $y^3 = e^{3x+6y}$

[5] (a) Find $\frac{dy}{dx}$ and show that it can be written in the form $\frac{dy}{dx} = \frac{y}{1-2y}$.

$$3y^2\frac{dy}{dx} = e^{3x+6y}\left(3+6\frac{dy}{dx}\right) = y^3\left(3+6\frac{dy}{dx}\right)$$

$$\frac{dy}{dx}(3y^2-6y^3) = 3y^3 \qquad \frac{dy}{dx} = \frac{y^3}{y^2-2y^3}$$

$$\boxed{\frac{dy}{dx} = \frac{y}{1-2y}}$$

since e^{3x+6y} cannot be zero $\Rightarrow y^3 \neq 0$
$\Rightarrow y \neq 0$.

[5] (b) Find $\frac{d^2y}{dx^2}$ in terms of y only.

$$\frac{d^2y}{dx^2} = \frac{(1-2y)\frac{dy}{dx} - y\left(-2\frac{dy}{dx}\right)}{(1-2y)^2} = \frac{1}{(1-2y)^2}\frac{dy}{dx}$$

$$\boxed{\frac{d^2y}{dx^2} = \frac{y}{(1-2y)^3}}$$

[5] (c) Find the equation of the tangent line to $y^3 = e^{3x+6y}$ at the point $(-2, 1)$.

At $(-2,1)$, $\frac{dy}{dx} = \frac{1}{1-2} = -1$

So $y-1 = -1(x+2)$ is the eqn of the tan line

$y-1 = -x-2$

$y = -x-1$

or $x+y = -1$

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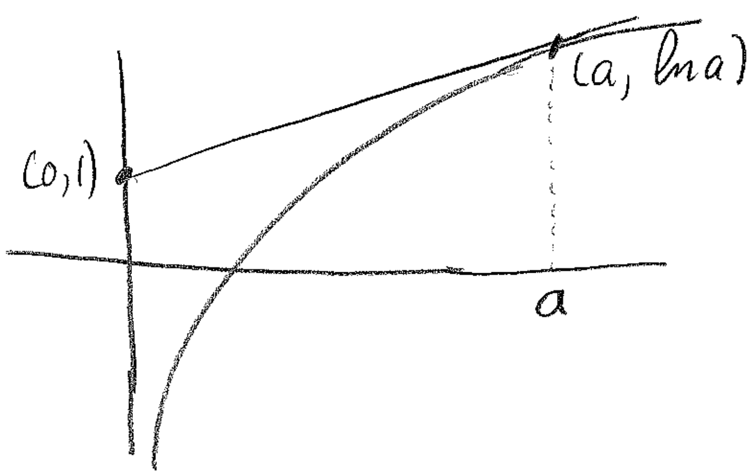
4. *[13 marks]*

Find the equation of the line that goes through the point $(x, y) = (0, 1)$ and which is also tangent to the graph of

$$y = \ln x$$

At what point(s) (x, y) does this tangent line intersect the graph of $y = \ln x$?



Let $(a, \ln a)$ be the point at which the tan line intersects the graph.

Since $y' = \frac{1}{x}$, the slope of the tan line is $\frac{1}{a}$.
But the slope is also $\frac{\ln a - 1}{a - 0}$. Hence

$$\frac{\ln a - 1}{a} = \frac{1}{a}$$

$$\ln a - 1 = 1$$

$$\ln a = 2$$

$$a = e^2$$

The point of intersection is $\boxed{(e^2, 2)}$

The eqn of the line is $y - 1 = \frac{1}{e^2}(x - 0)$

or $\boxed{y = \frac{1}{e^2} x + 1}$

Alternatively, $y - 2 = \frac{1}{e^2}(x - e^2)$ or $y = \frac{x}{e^2} + 1$