Solved

Department of Mathematics
University of Toronto

TUESDAY, December 14, 2004 9:00-11:00 AM
MAT 133Y TERM TEST #2

Calculus and Linear Algebra for Commerce
Duration: 2 hours

Aids Allowed: A non-graphing calculator, with empty memory, to be supplied by student.

Instructions: Fill in the information on this page, and make sure your test booklet contains 11 pages. In addition, you should have a **multiple-choice answer sheet**, on which you should fill in your name, number, tutorial time, tutorial room, and tutor's name.

This test consists of 10 multiple choice questions, and 4 written-answer questions.
For the **multiple choice questions** you can do your rough work in the test booklet, but you must record your answer by circling the appropriate letter **on the answer sheet** with your pencil. Each correct answer is worth 4 marks; a question left blank, or an incorrect answer, or two answers for the same question is worth 0. For the **written-answer questions**, present your solutions in the space provided. The value of each written-answer question is indicated beside it.
ENCLOSE YOUR FINAL ANSWER IN A BOX AND WRITE IT IN INK.

TOTAL MARKS: 100

FAMILY NAME: ______________________

GIVEN NAME: ______________________

STUDENT NO: ______________________

SIGNATURE: ______________________

TUTORIAL TIME and ROOM: ______________________

REGCODE and TIMECODE: ______________________

T.A.'S NAME: ______________________

Regcode	Timecode	Room	**Regcode**	Timecode	Room
T0101A	M9A	BF 323	T0501C	W3C	RW 229
T0101B	M9B	LM 123	T0501D	W3D	BA1240
T0101C	M9C	LM 157	T0601A	R4A	LM 157
T0201A	M3A	UCA101	T0701A	F2A	BF 323
T0201B	M3B	SS2106	T0701B	F2B	LM 157
T0201C	M3C	WB 119	T0701C	F2C	MP 118
T0201D	M3D	LM 123	T0801A	F3A	WA 142
T0301A	T3A	MP 137	T0801B	F3B	WI 523
T0301B	T3B	UC 328	T5101A	R5A	LM 155
T0301C	T3C	MP 134	T5101B	R5B	LM 157
T0401A	W9A	BF 323	T5201A	R6A	SS2111
T0401B	W9B	LM 123			
T0501A	W3A	LM 123			
T0501B	W3B	UC 328			

FOR MARKER ONLY	
Multiple Choice	
B1	
B2	
B3	
B4	
TOTAL	

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PART A. Multiple Choice

1. *[4 marks]*

   The derivative of $f(x) = \dfrac{x^2+1}{e^x+2}$ is:

   A. $\dfrac{2x}{e^x+2}$

   B. $\dfrac{2x}{e^x} - \dfrac{x^2+1}{e^x+2}$

   C. $\dfrac{2x(e^x+2) - e^x(x^2+1)}{(e^x+2)^2}$

   D. $\dfrac{(e^x+2)(2x-e^x)}{(e^x+2)^2}$

   E. $\dfrac{(2x+1)\cdot xe^x - (x^2+1)(e^x+2)}{(e^x+2)^2}$

$f'(x) =$

$$\frac{(e^x+2)(2x) - (x^2+1)e^x}{(e^x+2)^2}$$

2. *[4 marks]*

   Suppose we know that the function $f(x)$ does not have a maximum value on the interval $0 \le x \le 3$. Then we can be sure that:

   A. $f(x) = \dfrac{1}{x}$ Could be many other fcns, e.g. $f(x) = \frac{1}{x^2}$ (actually, $\frac{1}{x}$ not defined on $[0,3]$)

   B. $f(x)$ is not continuous on $[0,3]$

   C. The derivative of $f(x)$ at $x=1$ does not exist No. See example in E.

   D. The interval $0 \le x \le 3$ is not closed It *is* closed.

   E. $f(x)$ gets bigger than any given number on this interval.

   No. $f(x)$ could be $f(x) = \begin{cases} x & 0 \le x < 3 \\ 0 & x = 3 \end{cases}$

B. EVT says: a cont. fcn. on a closed and bounded interval *must* have a max (and a min.)

Since $[0,3]$ is closed and bounded, and f does *not* have a max, it can't be cont.

3. *[4 marks]*

$$\lim_{x \to -\infty} \frac{4 - 3e^{-x}}{1 + e^{-x}}$$

A. equals -3

B. equals -1

C. equals 4

D. equals 3

E. does not exist

$$= \lim_{x \to -\infty} \frac{e^{-x}(4e^{x} - 3)}{e^{-x}(e^{x} + 1)} = \lim_{x \to -\infty} \frac{4e^{x} - 3}{e^{x} + 1} = -3$$

since $\lim_{x \to -\infty} e^{x} = 0$.

4. *[4 marks]*

$$\lim_{x \to 2} \frac{\sqrt{x^2 - 1} - \sqrt{x + 1}}{x - 2}$$

A. equals 1

B. equals $\frac{\sqrt{3}}{2}$

C. equals 3

D. equals 0

E. does not exist

$$= \lim_{x \to 2} \frac{(x^2 - 1) - (x + 1)}{(x-2)(\sqrt{x^2 - 1} + \sqrt{x+1})}$$

$$= \lim_{x \to 2} \frac{x^2 - x - 2}{(x-2)(\sqrt{x^2 - 1} + \sqrt{x+1})}$$

$$= \lim_{x \to 2} \frac{x + 1}{\sqrt{x^2 - 1} + \sqrt{x+1}}$$

$$= \frac{3}{2\sqrt{3}} = \frac{\sqrt{3}}{2}$$

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5. *[4 marks]*

The annual rate r compounded continuously which is equivalent to 6% compounded quarterly is closest to:

A. 5.83%

B. 6.05%

C. 5.96%

D. 5.9%

E. 6.14%

$$e^r = \left(1 + \frac{.06}{4}\right)^4$$

$$r = 4 \ln 1.015$$

$$= .059554$$

$$\approx .0596$$

6. *[4 marks]*

The solution of the inequality $\dfrac{x^2 + 3x - 4}{4 - x^2} \geq 0$ is given by:

A. $-4 < x < -2$ or $1 < x < 2$

B. $x \leq -4$ or $-2 < x \leq 1$ or $x > 2$

C. $-4 \leq x \leq -2$ or $1 \leq x \leq 2$

D. $x \leq -4$ or $-2 \leq x \leq 1$ or $x \geq 2$

E. $-4 \leq x < -2$ or $1 \leq x < 2$

$$\frac{(x+4)(x-1)}{(2-x)(2+x)}$$

Int	f
$(-\infty,-4)$	$-$
$(-4,-2)$	$+$
$(-2,1)$	$-$
$(1,2)$	$+$
$(2,\infty)$	$-$

$f > 0$ on $(-4,-2)$ and $(1,2)$ and $f = 0$ at 1 and -4.

$f \geq 0$ on $[-4,-2)$ and also on $[1,2)$

$-4 \leq x < -2$ $\quad$ $1 \leq x < 2$

7. *[4 marks]*

Given that $f(0) = 2$ and the slope of the tangent line to the graph of $f(x)$ at the point $(0, 2)$ equals 3, the slope of the tangent line to the graph of $\left(f(x)\right)^5$ at $x = 0$ equals

A. 80

B. 32

C. 240

D. 3^5

E. 3^4

$$y = [f(x)]^5$$
$$y' = 5[f(x)]^4 f'(x)$$
$$y'(0) = 5[f(0)]^4 f'(0)$$
$$= 5 \cdot 2^4 \cdot 3 = 240$$

8. *[4 marks]*

If $y = y(x)$ satisfies $y^x = e^y$, then when $(x, y) = (2\sqrt{e}, \sqrt{e})$, $\dfrac{dy}{dx} =$

A. $-\dfrac{1}{2}$

B. $-\dfrac{1}{\sqrt{e}}$

C. 2

D. $\sqrt{e}$

E. $\dfrac{\sqrt{e}}{2}$

$$x \ln y = y$$
$$\ln y + \frac{x}{y} y' = y'$$
$$\ln \sqrt{e} + 2y' = y'$$
$$\frac{1}{2} \ln e = -y'$$
$$-\frac{1}{2} = y'$$

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9. *[4 marks]*

If $x_1 \neq 1$ is used as an initial estimate for a solution of $f(x) = 0$, when $f(x) = \dfrac{x}{1-x}$, then Newton's method yields the second estimate $x_2 =$

A. $\sqrt{x_1}$

B. 0

C. $2x_1$

D. x_1^2

E. $\frac{1}{2}x_1$

$$f'(x) = \frac{1}{(1-x)^2}$$

$$x_2 = x_1 - \frac{\frac{x_1}{1-x_1}}{\frac{1}{(1-x_1)^2}}$$

$$= x_1 - x_1(1-x_1) = x_1^2$$

10. *[4 marks]*

If $y(x) = (x+1)\,(\frac{1}{2}x+1)^2\,(\frac{1}{3}x+1)^3\,(\frac{1}{4}x+1)^4\,(\frac{1}{5}x+1)^5$, then $y'(0) =$

A. $\frac{1}{1}\cdot\frac{1}{2^2}\cdot\frac{1}{3^3}\cdot\frac{1}{4^4}\cdot\frac{1}{5^5}$

B. 15

C. 5

D. $1\cdot 2^2\cdot 3^3\cdot 4^4\cdot 5^5$

E. 120

$$\ln y = \ln(x+1) + 2\ln\left(\frac{x}{2}+1\right) + 3\ln\left(\frac{x}{3}+1\right) + 4\ln\left(\frac{x}{4}+1\right) + 5\ln\left(\frac{x}{5}+1\right)$$

$$\frac{1}{y}y' = \frac{1}{x+1} + \frac{1}{\frac{x}{2}+1} + \frac{1}{\frac{x}{3}+1} + \frac{1}{\frac{x}{4}+1} + \frac{1}{\frac{x}{5}+1}$$

and since $y(0) = 1$,

$$y'(0) = 5$$

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PART B. Written-Answer Questions

1. *[14 marks]*
Given

$$f(x) = \begin{cases} 5^{\frac{1}{x}} & \text{if } x < 0 \\ \dfrac{x^2 - 2x}{x - 2} & \text{if } 0 \le x < 2 \\ \dfrac{4b}{x} & \text{if } x \ge 2, \text{ where } b \text{ is a constant.} \end{cases}$$

Showing your steps clearly

[7] (a) determine whether f is continuous at $x = 0$.

$\lim_{x\to 0^-} f(x) = \lim_{x\to 0^-} 5^{\frac{1}{x}} = 0$. $f(0) = \dfrac{0^2 - 2\cdot 0}{0-2} = 0$

So far so good.

$\lim_{x\to 0^+} f(x) = \lim_{x\to 0^+} \dfrac{x^2-2x}{x-2} = \lim_{x\to 0^+} x = 0$

$\therefore \lim_{x\to 0} f(x) = 0 = f(0)$ and $\boxed{f \text{ is cont at } 0.}$

[7] (b) determine whether there is a value of the constant b that makes f continuous at $x = 2$.

$\lim_{x\to 2^-} f(x) = \lim_{x\to 2^-} \dfrac{x^2-2x}{x-2} = \lim_{x\to 2^-} x = 2$

$\lim_{x\to 2^+} f(x) = \lim_{x\to 2^+} \dfrac{4b}{x} = 2b$

So $\lim_{x\to 2} f(x)$ will exist only if $b = 1$

and then $f(2) = 2 = \lim_{x\to 2} f(x)$

so $\boxed{b = 1 \text{ makes } f \text{ cont at } 2}$

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2. *[12 marks]*

Unit price p and demand quantity q for a certain good are related by the equation

$$q = \left(1000 - \frac{1}{2}p\right) e^{-\frac{p}{500}}$$

where p is in the range $\$500 \le p \le \$700$.

Find the marginal revenue $\frac{dr}{dq}$ where p is \$600.

$r = pq \qquad MR = \frac{dr}{dq} = p + q\frac{dp}{dq}$

When $p = 600$, $q = 700e^{-\frac{6}{5}}$. So $\left.\frac{dr}{dq}\right|_{p=600} =$

$600 + 700e^{-\frac{6}{5}} \left.\frac{dp}{dq}\right|_{p=600}$. To find $\frac{dp}{dq}$,

$1 = -\frac{1}{2}\frac{dp}{dq}e^{-\frac{p}{500}} + \left(1000 - \frac{1}{2}p\right)e^{-\frac{p}{500}} \cdot \left(-\frac{1}{500}\right)\frac{dp}{dq}$

$1 = e^{-\frac{p}{500}} \frac{dp}{dq}\left(-\frac{1}{2} - 2 + \frac{p}{1000}\right)$. Substituting $p = 600$,

$1 = e^{-\frac{6}{5}} \left.\frac{dp}{dq}\right|_{p=600} \left(-\frac{19}{10}\right)$ and $\left.\frac{dp}{dq}\right|_{p=600} = -\frac{10}{19}e^{\frac{6}{5}}$

$$\left.\frac{dr}{dq}\right|_{p=600} = 600 + 700e^{-\frac{6}{5}}e^{\frac{6}{5}}\left(-\frac{10}{19}\right)$$

$$= 600 - \frac{7000}{19}$$

$$\approx \boxed{\$\ 231.58}$$

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3. *[16 marks]*

If a demand curve is given by

$$q^2 + qp + 2p^2 = 1100$$

[8] (a) What is the (point) elasticity of demand when $q = 10$, $p = 20$?

$\mu = \frac{p}{q}\frac{dq}{dp}$

$$2q\frac{dq}{dp} + q + p\frac{dq}{dp} + 4p = 0$$

$$\frac{dq}{dp} = -\frac{(4p+q)}{(p+2q)} = -\frac{90}{40} \text{ at } \begin{matrix} q=10 \\ p=20 \end{matrix}$$

$$\mu = \frac{20}{10}\left(-\frac{90}{40}\right) = \boxed{-\frac{9}{2} = -4.5}$$

[8] (b) At the same point, $q = 10$, $p = 20$, is the demand function $p(q)$ concave up or concave down? Justify your answer. [This is **not** an invitation for an essay on demand curves; the question is about **this** demand curve only.]

We could, more efficiently for doing part b), have done part (a) by using $\mu = \frac{p}{q\frac{dp}{dq}}$ and finding $\frac{dp}{dq} = -\frac{(p+2q)}{(4p+q)}$

and $\frac{dp}{dq} = -\frac{4}{9}$ at $q=10$, $p=20$

Then $$\frac{d^2p}{dq^2} = -\left[\frac{(4p+q)\left[\frac{dp}{dq}+2\right] - (p+2q)\left[4\frac{dp}{dq}+1\right]}{(4p+q)^2}\right]$$

At $\begin{matrix} q=10 \\ p=10 \end{matrix}$ $$\frac{d^2p}{dq^2} = -\left[\frac{90\left(-\frac{4}{9}+2\right) - 40\left(-\frac{16}{9}+1\right)}{90^2}\right]$$

$$= -\left(\frac{154}{7290}\right) < 0 \quad (\sim -.02)$$

So $\boxed{p(q) \text{ is concave down}}$

$\frac{d^2p}{dq^2}$ can also be calculated by starting with $2q + p + q\frac{dp}{dq} + 4p\frac{dp}{dq} = 0$, then differentiating again.

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4. *[18 marks]*

Given $$f(x) = \frac{2x^3}{1-x^2}$$

so that $$f'(x) = \frac{2x^2(3-x^2)}{(1-x^2)^2}$$

and $$f''(x) = \frac{4x(x^2+3)}{(1-x^2)^3}$$

then, on the way to sketching the graph of $y = f(x)$, justifying your answers,

(a) find all vertical and horizontal aysmptotes if any

(b) find where f is increasing, decreasing and all relative extrema if any

(c) find where f is concave up, concave down and all inflection points if any

(d) Now sketch $y = f(x)$, showing all the features in (a), (b) and (c).

a) $\boxed{\text{V.A. at } x=-1 \text{ and } x=1}$ $\lim_{x\to\infty} f(x) = -\infty$; $\lim_{x\to-\infty} f(x) = \infty$

so $\boxed{\text{no H.A.}}$

b)

Interval	f'	f
$(-\infty, -\sqrt{3})$	$-$	dec
$(-\sqrt{3}, -1)$	$+$	inc
$(-1, 0)$	$+$	inc
$(0, 1)$	$+$	inc
$(1, \sqrt{3})$	$+$	inc
$(\sqrt{3}, \infty)$	$-$	dec

$-\vee+$ $\boxed{\text{a relative min at } x = -\sqrt{3}}$

$+\wedge-$ a $\boxed{\text{relative max at } x = \sqrt{3}}$

c)

Interval	f''	f
$(-\infty, -1)$	$+$	conc up
$(-1, 0)$	$-$	conc down
$(0, 1)$	$+$	conc up
$(1, \infty)$	$-$	conc down

inflection at $x = 0$ only

$x = \pm 1$ not a point on the curve

sketch on next page

Question 4 continues on Page 11

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(BLANK SHEET FOR QUESTION 4)