Solved

Department of Mathematics
University of Toronto

TUESDAY, DECEMBER 9, 2003 9:00-11:00 AM

MAT 133Y TERM TEST #2

Calculus and Linear Algebra for Commerce

Duration: 2 hours

Aids Allowed: A non-graphing calculator, with empty memory, to be supplied by student.

Instructions: Fill in the information on this page, and make sure your test booklet contains 12 pages. In addition, you should have a **multiple-choice answer sheet**, on which you should fill in your name, number, tutorial time, tutorial room, and tutor's name.

This test consists of 10 multiple choice questions, and 4 written-answer questions.
For the **multiple choice questions** you can do your rough work in the test booklet, but you must record your answer by circling the appropriate letter **on the answer sheet** with your pencil. Each correct answer is worth 4 marks; a question left blank, or an incorrect answer, or two answers for the same question is worth 0. For the **written-answer questions**, present your solutions in the space provided. The value of each written-answer question is indicated beside it. ENCLOSE YOUR FINAL ANSWER IN A BOX AND WRITE IT IN INK.

TOTAL MARKS: 100

FAMILY NAME: ______________________

GIVEN NAME: ______________________

STUDENT NO: ______________________

SIGNATURE: ______________________

TUTORIAL TIME and ROOM: ______________________

REGCODE and TIMECODE: ______________________

T.A.'S NAME: ______________________

Regcode	Timecode	Room	**Regcode**	Timecode	Room
T0101A	M9A	SS2130	T0501C	W3C	SS2111
T0101B	M9B	BF 215	T0601A	R4A	SS1088
T0101C	M9C	LM 157	T0601B	R4B	RW 142
T0201A	M3A	SS2111	T0601C	R4C	LM 157
T0201B	M3B	LM 123	T0701A	F2A	BF 323
T0201C	M3C	RW 143	T0701B	F2B	LM 157
T0201D	M3D	MP 134	T0701C	F2C	SS2111
T0301A	T3A	RW 142	T0801A	F3A	RW 142
T0301B	T3B	SS2111	T0801B	F3B	LM 157
T0301C	T3C	MP 134	T5101A	R5A	SS1088
T0401A	W9A	BF 215	T5201A	R6A	SS1088
T0401B	W9B	LM 157			
T0501A	W3A	LM 157			
T0501B	W3B	LM 123			

FOR MARKER ONLY	
Multiple Choice	
B1	
B2	
B3	
B4	
TOTAL	

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PART A. Multiple Choice

1. *[4 marks]*

In the solution to the following system

$$\begin{array}{rcrcrl} 2x & + & y & - & z & = 7 \\ x & + & 3y & + & z & = 4 \\ 3x & + & y & + & z & = 2 \end{array}$$

$z =$

A. 2

B. -3

C. 1

D. 3

E. -2

$$z = \frac{\begin{vmatrix} 2 & 1 & 7 \\ 1 & 3 & 4 \\ 3 & 1 & 2 \end{vmatrix}}{\begin{vmatrix} 2 & 1 & -1 \\ 1 & 3 & 1 \\ 3 & 1 & 1 \end{vmatrix}} = \frac{\begin{vmatrix} 0 & -5 & -1 \\ 1 & 3 & 4 \\ 0 & -8 & -10 \end{vmatrix}}{\begin{vmatrix} 0 & -5 & -3 \\ 1 & 3 & 1 \\ 0 & -8 & -2 \end{vmatrix}} = \frac{-\begin{vmatrix} -5 & -1 \\ -8 & -10 \end{vmatrix}}{-\begin{vmatrix} -5 & -3 \\ -8 & -2 \end{vmatrix}}$$

$$= \frac{50-8}{10-24} = \frac{42}{-14} = \boxed{-3}$$

2. *[4 marks]*

Assume $p(x)$ and $q(x)$ are polynomials and that $\lim_{x\to -1} \frac{p(x)}{q(x)}$ exists and is equal to some nonzero value c. Then $\lim_{x\to -1} \frac{q(x)^2 + p(x)\,q(x)}{p(x)^2} =$

$$\lim_{x\to -1}\left[\frac{q(x)}{p(x)}\right]^2 + \lim_{x\to -1}\left[\frac{q(x)}{p(x)}\right]$$

$$= \boxed{\frac{1}{c^2} + \frac{1}{c}}$$

A. $c^2 + c$

B. $c^{-2} + c^{-1}$

C. $c^2 + c^{-1}$

D. $c^{-2} + c$

E. cannot be determined from the given information.

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3. *[4 marks]*

$\lim_{x\to\infty} \frac{(2x+1)\ (x-3)}{8x^2-5x+2} =$ $\lim_{x\to\infty} \frac{(2+\frac{1}{x})(1-\frac{3}{x})}{8-\frac{5}{x}+\frac{2}{x^2}} = \frac{2}{8} = \boxed{\frac{1}{4}}$

A. Undefined

B. ∞

C. $-3/2$

D. 0

E. $1/4$

4. *[4 marks]*

$\lim_{x\to -1^-} 5 - 3^{\frac{1}{1-x^2}} =$

A. 2

B. 4

C. $+\infty$

D. 5

E. $-\infty$

$x < -1$

$x^2 > 1$

$1 - x^2 < 0$

$\frac{1}{1-x^2} \to -\infty$ as $x \to -1^-$

$3^{\frac{1}{1-x^2}} \to 0$ as $x \to -1^-$

$\lim_{x\to -1^-} 5 - 3^{\frac{1}{1-x^2}} = 5$

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5. *[4 marks]*

$\dfrac{(x^2-1)^2}{x} > 0$ precisely when

$(x^2-1)^2 > 0, \; x \neq \pm 1$

So $\dfrac{(x^2-1)^2}{x} > 0 \quad x > 0$

A. $-\infty < x < -1$ or $0 < x < 1$

B. $-1 < x < 0$ or $1 < x < \infty$

C. $-1 < x < 1$

D. $x < -1$ or $x > 1$

E. $x > 0$ and $x \neq 1$

6. *[4 marks]*

If $y = (1-x)\,(2-x)^2\,(3-x)^3$ then $y'(0) =$

A. 108

B. -243

C. -324

D. -81

E. -108

$\ln y = \ln(1-x) + 2\ln(2-x) + 3\ln(3-x)$

$\frac{1}{y} y' = -\frac{1}{1-x} - \frac{2}{2-x} - \frac{3}{3-x}$

$y(0) = 4 \times 27$, so $y'(0) = 108\left[-1-1-1\right] = \boxed{-324}$

or $y' = -(2-x)^2(3-x)^3 - 2(1-x)(2-x)(3-x)^3 - 3(1-x)(2-x)^2 \times (3-x)^2$

$y'(0) = -4\cdot 27 - 2\cdot 2\cdot 27 - 3\cdot 4\cdot 9 = \boxed{-324}$

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7. *[4 marks]*

The slope of the tangent to the graph of $y = \sqrt{2e^x + \sqrt{3e^x + e^{-x}}}$ at the point $(0,\ 2)$ is

A. $\frac{3}{4}$

B. $\frac{5}{8}$

C. 0

D. 2

E. $\frac{5}{2}$

$$y' = \frac{1}{2\sqrt{2e^x + \sqrt{3e^x + e^{-x}}}} \cdot \left(2e^x + \frac{3e^x - e^{-x}}{2\sqrt{3e^x + e^{-x}}}\right)$$

$$y'(0) = \frac{1}{2\sqrt{2+\sqrt{4}}}\left(2 + \frac{2}{2\sqrt{4}}\right)$$

$$= \frac{1}{2 \cdot 2}\left(2 + \frac{1}{2}\right) = \boxed{\frac{5}{8}}$$

8. *[4 marks]*

If $y^5 = 32x + y \ln x$ then when $x = 1$ and $y = 2$, $\frac{dy}{dx} =$

A. $\frac{17}{40}$

B. $\frac{17}{10}$

C. $-\frac{27}{40}$

D. $-\frac{17}{10}$

E. $\frac{27}{20}$

$$5y^4 y' = 32 + y' \ln x + \frac{y}{x}$$

$$5 \cdot 16\, y' = 32 + 2$$

$$y' = \frac{34}{80} = \frac{17}{40}$$

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9. *[4 marks]*

If $f(8) = 2$ and $f'(8) = 3$, then $\lim_{h \to 0} \dfrac{\ln(f(8+h)) - \ln 2}{h} = \lim_{h \to 0} \dfrac{\ln(f(8+h)) - \ln(f(8))}{2}$

A. $\ln 3$

B. 6

C. $\frac{3}{2}$

D. $\frac{\ln 3}{2}$

E. 3

$= \frac{d}{dx} \ln(f(x))$ at $x = 8$.

$= \frac{f'(x)}{f(x)}$ at $x = 8$

$= \frac{f'(8)}{f(8)} = \boxed{\frac{3}{2}}$

10. *[4 marks]*

If a revenue function is given, in dollars, by

$$r(q) = (100 - 2\ln q)\, q$$

$\frac{dr}{dq} = (100 - 2\ln q) - \frac{2}{q} \cdot q$

and a production function is given by

$$q = 10(\sqrt{m} - 5)$$

where m is the number of employees and q is the quantity produced, then, when $m = 100$, the marginal revenue product is closest to

A. \$230,440

B. \$4,609

C. \$115,220

D. \$2,304

E. \$45

$\frac{dr}{dm} = \frac{dr}{dq} \frac{dq}{dm} = (98 - 2\ln q)\left(\frac{5}{\sqrt{m}}\right)$

When $m = 100$ $q = 10 \cdot 5 = 50$.

$\left.\frac{dr}{dm}\right|_{m=100} = (98 - 2\ln 50)\frac{1}{2}$

≈ 45.09

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PART B. Written-Answer Questions

1. *[16 marks]*

(a) *[7 marks]*

Evaluate $\lim_{t\to 2} \frac{|t-2|}{|t|-2}$ or show that it doesn't exist.

Hint: You may want to consider each of the one sided limits first.

When $t \to 2$, $t > 0$; so $|t| = t$

$$\lim_{t\to 2^+} \frac{|t-2|}{|t|-2} = \lim_{t\to 2^+} \frac{t-2}{t-2} = 1$$

$$\lim_{t\to 2^-} \frac{|t-2|}{|t|-2} = \lim_{t\to 2^-} \frac{-(t-2)}{t-2} = -1$$

Since the limit from the right and the limit from the left are not equal, the limit itself $\boxed{\text{does not exist}}$.

Saying that $\frac{0}{0}$ is not defined and "hence" the limit does not exist is worth NOTHING!

1. (b) *[9 marks]*

For what values of b and c is the function

$$f(x) = \begin{cases} \dfrac{x^4 + bx^3 - 10x^2}{(x+c)(x-2)} & x > 2 \\ 4 & x \le 2. \end{cases}$$

continuous for all values of x? Make sure to explain your answer properly.

For $x>2$, $f(x) = \dfrac{x^2(x^2+bx-10)}{(x+c)(x-2)}$

For f to even have a limit from the right at $x=2$, $x-2$ must be a factor of $x^2+bx-10 = (x-2)(x-r)$

r must be -5 to match the -10.

$(x-2)(x+5) = x^2+3x-10$ so $\boxed{b=3}$

$$f(x) = \begin{cases} \dfrac{x^2(x+5)}{x+c} & x > 2 \\ 4 & x \le 2 \end{cases}$$

$$\lim_{x \to 2^+} f(x) = \frac{28}{2+c}$$

$$\lim_{x \to 2^-} f(x) = 4 = f(2) \text{ by definition}$$

For continuity everywhere, all that is now required is $\dfrac{28}{2+c} = 4$ so $\boxed{c=5}$

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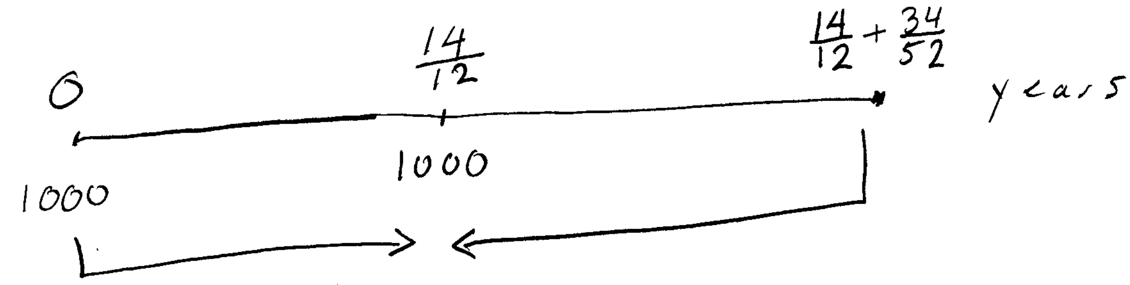
2. *[11 marks]*

Three payments of \$1,000 each are to be made into an account which earns 10% nominal annual interest compounded continuously. The second payment is to be made 14 months after the initial payment, and the third payment is to be made 34 weeks after the second payment. Find the total value of all three payments calculated at the time the second payment is made. [You may consider each month to be $\frac{1}{12}$ of a year and each week $\frac{1}{52}$ of a year.]



$$V = 1000e^{.10\left(\frac{14}{12}\right)} + 1000 + 1000e^{-.10\left(\frac{34}{52}\right)}$$

$$= \boxed{\$\,3060.45}$$

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3. *[18 marks]*

Given the input-output matrix for the three industries

	Industry A	Industry B	Industry C		Final Demand
Industry A	40	0	25	:	135
Industry B	80	60	25	:	135
Industry C	0	120	75	:	55
Other Production Factors	80	120	125	:	–

[6] (a) Find the coefficient matrix (technology matrix).

[9] (b) Find the new output for each of the industries if final demand changes to 100 for each of the industries.

[3] (c) How much, in total, of the "Other Production Factors" is used to meet the new final demand?

a) Row A has inputs from industry A to itself and others
= total output of industry A $= 40+25+135 = 200$

Similarly, " " " " $B = 80+60+25+135 = 300$

" " " " $C = 120+75+55 = 250$

Col A has inputs from all industries to industry A.
Dividing by total output of industry A gives input from each industry per unit out of industry A.
Similarly Col B and Col C for industries B and C

$$A = \begin{pmatrix} \frac{40}{200} & 0 & \frac{25}{250} \\ \frac{80}{200} & \frac{60}{300} & \frac{25}{250} \\ 0 & \frac{120}{300} & \frac{75}{250} \end{pmatrix} = \boxed{\begin{pmatrix} \frac{1}{5} & 0 & \frac{1}{10} \\ \frac{2}{5} & \frac{1}{5} & \frac{1}{10} \\ 0 & \frac{2}{5} & \frac{3}{10} \end{pmatrix}}$$

Note: for part (c) we will also need to know

$\frac{80}{200} = \frac{2}{5}$ unit of "other" per unit of A

$\frac{120}{300} = \frac{2}{5}$ " " " " of B

$\frac{125}{250} = \frac{1}{2}$ " " " " of C

Question 3 continues on Page 11

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(BLANK SHEET FOR QUESTION 3)

3b) $I - A = \begin{pmatrix} \frac{4}{5} & 0 & -\frac{1}{10} \\ -\frac{2}{5} & \frac{4}{5} & -\frac{1}{10} \\ 0 & -\frac{2}{5} & \frac{7}{10} \end{pmatrix}$. Solve $(I-A)X = \begin{pmatrix} 100 \\ 100 \\ 100 \end{pmatrix}$

where $X = \begin{pmatrix} x_A \\ x_B \\ x_C \end{pmatrix}$

Multiplying by 10,

$$\left(\begin{array}{ccc|c} 8 & 0 & -1 & 1000 \\ -4 & 8 & -1 & 1000 \\ 0 & -4 & 7 & 1000 \end{array}\right) \begin{array}{l} R_1 \leftrightarrow R_2 \\ R_2 \to R_2 + 2R_1 \\ R_1 \to -\frac{1}{4}R_1 \end{array} \left(\begin{array}{ccc|c} 1 & -2 & \frac{1}{4} & -250 \\ 0 & 16 & -3 & 3000 \\ 0 & -4 & 7 & 1000 \end{array}\right)$$

$$\begin{array}{l} \xrightarrow{R_2 \leftrightarrow R_3} \\ R_2 \to -\frac{1}{4}R_2 \\ R_3 \to R_3 - 16R_2 \end{array} \left(\begin{array}{ccc|c} 1 & -2 & \frac{1}{4} & -250 \\ 0 & 1 & -\frac{7}{4} & -250 \\ 0 & 0 & 25 & 7000 \end{array}\right) \longrightarrow \left(\begin{array}{ccc|c} 1 & -2 & \frac{1}{4} & -250 \\ 0 & 1 & -\frac{7}{4} & -250 \\ 0 & 0 & 1 & 280 \end{array}\right)$$

$x_C = 280$

$x_B = -250 + \frac{7}{4}x_C = -250 + 490 = 240$

$x_A = -250 - \frac{1}{4}x_C + 2x_B = -250 - 70 + 480 = 160$

$$\boxed{X = \begin{pmatrix} x_A \\ x_B \\ x_C \end{pmatrix} = \begin{pmatrix} 160 \\ 240 \\ 280 \end{pmatrix}}$$

3c) From Note at the end of (a),

Other production factors $= \frac{2}{5} \times 160 + \frac{2}{5} \times 240 + \frac{1}{2} \times 280$

$= \boxed{300}$

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4. *[15 marks]*

The Powdered Rum Company is a monopoly which sells its product to navies the world over. When price (p) is given in dollars per gram and demand (q) is given in grams per second the demand for its product satisfies

$$2p^2 + 2pq + q^2 = 1000.$$

Note that $q = 20$ when $p = 10$.

[7] (a) Find $\frac{dq}{dp}$ when $p = 10$.

$$4p + 2q + 2p\frac{dq}{dp} + 2q\frac{dq}{dp} = 0 \Rightarrow 40 + 40 + 20\frac{dq}{dp} + 40\frac{dq}{dp} = 0$$

$$\frac{dq}{dp} = -\frac{80}{60} = \boxed{-\frac{4}{3}}$$

or

$$\frac{dq}{dp} = -\frac{2p+q}{p+q} = \boxed{-\frac{4}{3}} \text{ at } q = 20,\ p = 10$$

[6] (b) Find marginal revenue *with respect to price* when $p = 10$.
That is, find $\frac{dr}{dp}$ when $p = 10$, where r denotes revenue in dollars per second.

$$r = pq$$

$$\frac{dr}{dp} = q + p\frac{dq}{dp}$$

$$= 20 + 10\left(-\frac{4}{3}\right) \quad \text{when } p = 10, \text{ and } q = 20.$$

$$\boxed{\frac{dr}{dp} = \frac{20}{3} \approx \$6.67}$$

[2] (c) Find the approximate change in revenue which would result if the Company were to raise its price from 10 to 12 dollars per gram.

$$\Delta r \approx \frac{dr}{dp}\Delta p = \frac{20}{3} \times 2 = \boxed{\frac{40}{3} \approx \$13.33}$$