Solved

Department of Mathematics
University of Toronto

WEDNESDAY, DECEMBER 11, 2002
MAT 133Y TERM TEST #2

Calculus and Linear Algebra for Commerce
Duration: 1 hour 50 minutes

Aids Allowed: A non-graphing calculator, with empty memory, to be supplied by student.

Instructions: Fill in the information on this page, and make sure your test booklet contains 11 pages. In addition, you should have a **multiple-choice answer sheet**, on which you should fill in your name, number, tutorial time, tutorial room, and tutor's name.

This test consists of 10 multiple choice questions, and 4 written-answer questions.
For the **multiple choice questions** you can do your rough work in the test booklet, but you must record your answer by circling the appropriate letter **on the answer sheet** with your pencil. Each correct answer is worth 4 marks; a question left blank, or an incorrect answer, or two answers for the same question is worth 0. For the **written-answer questions** , present your solutions in the space provided. The value of each written-answer question is indicated beside it.
ENCLOSE YOUR FINAL ANSWER IN A BOX AND WRITE IT IN INK.

TOTAL MARKS: 100

FAMILY NAME: ____________________

GIVEN NAME: ____________________

STUDENT NO: ____________________

SIGNATURE: ____________________

TUTORIAL TIME and ROOM: ____________________

REGCODE and TIMECODE: ____________________

T.A.'S NAME: ____________________

Regcode	Timecode	Room	**Regcode**	Timecode	Room
T0101A	M9A	RW142	T0501C	W3C	LA341
T0101B	M9B	LM157	T0501D	W3D	NF 6
T0101C	M9C	LM123	T0601A	R4A	RW142
T0201A	M3A	LM157	T0601B	R4B	UC244
T0201B	M3B	UC85	T0601C	R4C	SS2130
T0201C	M3C	LA240	T0701A	F2A	MP118
T0201D	M3D	LA204	T0701B	F2B	SS2130
T0301A	T3A	VC212	T0701C	F2C	RW229
T0301B	T3B	NF113	T0801A	F3A	RW142
T0301C	T3C	CR403	T0801B	F3B	SS2111
T0301D	T3D	NF 7	T5101A	R5A	MP118
T0401A	W9A	LM157	T5101C	R5C	UC244
T0401B	W9B	MP118	T5201B	R6B	LM157
T0401C	W9C	LM155			
T0501A	W3A	RW143			
T0501B	W3B	RW229			

FOR MARKER ONLY	
Multiple Choice	
B1	
B2	
B3	
B4	
TOTAL	

NAME:______________________________ STUDENT NO:______________________________

PART A. Multiple Choice

1. *[4 marks]* $\lim_{x\to+\infty}\left(1+\frac{3}{x}\right)^x = \lim_{y\to\infty}\left[\left(1+\frac{1}{y}\right)^y\right]^3 = e^3$

Let $y = \frac{x}{3}$

$x \to \infty \Rightarrow y \to \infty$

A. 3

B. e

C. e^3

D. $+\infty$

E. 1

2. *[4 marks]* $\lim_{x\to+\infty} x\left(\sqrt{x^2+5}-\sqrt{x^2+2}\right) = \lim_{x\to\infty} \frac{x\,(3)}{\sqrt{x^2+5}+\sqrt{x^2+2}}$

A. 0

B. 1

C. 3/2

D. 7/2

E. $+\infty$

$$= \lim_{x\to\infty} \frac{3x}{x\left(\sqrt{1+\frac{5}{x^2}}+\sqrt{1+\frac{2}{x^2}}\right)}$$

$$= \lim_{x\to\infty} \frac{3}{\sqrt{1+\frac{5}{x^2}}+\sqrt{1+\frac{2}{x^2}}}$$

$$= \frac{3}{2}$$

3. *[4 marks]* If \$500 is invested at 8% compounded continuously then the amount (to the nearest cent) at the end of 15 years is

   $S = 500e^{.08 \times 15} = \$1660.06$

   A. \$1100.00

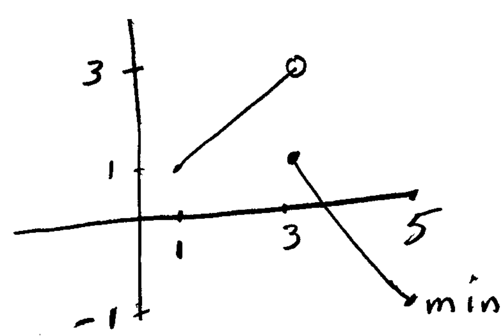
   B. \$13,576.06

   C. \$150.60

   D. \$1586.08

   E. \$1660.06

4. *[4 marks]* If $f(x) = \begin{cases} x & 1 \le x < 3 \\ 4 - x & 3 \le x \le 5 \end{cases}$
   then, on the interval $1 \le x \le 5$,



no max

   A. f has neither a maximum nor a minimum

   B. f has a maximum and a minimum

   C. f has a maximum but no minimum

   D. f has a minimum but no maximum

   E. f is not a function

NAME:____________ STUDENT NO:____________

5. *[4 marks]* If $y = ax + b$ is the equation of the line which is tangent, at $(x, y) = (e, e)$, to the graph of $y = x \ln x$, then $ab =$

   A. $-2e^2$

   B. $-e$

   C. $2e^2$

   D. e

   E. $-2e$

$y - e = m(x - e)$ where $m = y'(e)$

$y' = \ln x + 1 \qquad y'(e) = \ln e + 1 = 2$

$y - e = 2(x - e)$

$y = 2x - e \qquad a = 2 \quad b = -e$

$ab = -2e$

6. *[4 marks]* Suppose $f(x)$ and $g(x)$ are functions which satisfy:

$$f(2) = 3, \quad g(2) = 5$$
$$f'(2) = 7, \quad g'(2) = 11$$
$$f'(3) = 13, \quad g'(3) = 17$$
$$f'(5) = 19, \quad g'(5) = 23$$

   If $h(x) = g(f(x))$ then $h'(2) =$ $g'(f(2))(f'(2))$

$= g'(3) f'(2)$

$= 17 \cdot 7 = 119$

   A. 161

   B. 143

   C. 209

   D. 119

   E. 77

NAME:________________ STUDENT NO:________________

7. *[4 marks]* Let $y = \dfrac{8\sqrt{(4-x^2)^3}}{(x+1)^{10}(x+2)^5}$. Then at $x = 0$, $y' =$

A. 2

B. 20

C. −25

D. −30

E. −32

$$\ln y = \ln 8 + \frac{3}{2}\ln(4-x^2) - 10\ln(x+1) - 5\ln(x+2)$$

$$\frac{1}{y}y' = \frac{3\cdot(-2x)}{2(4-x^2)} - \frac{10}{x+1} - \frac{5}{x+2}$$

At $x=0$, $y = \dfrac{8\sqrt{4^3}}{2^5} = 2$

So $\frac{1}{2}y' = -10 - \frac{5}{2} = -\frac{25}{2}$

$y' = -25$

8. *[4 marks]* The equation $y^2 + 4x^2 = 3xy + x + 9$ defines y implicitly as a function of x near the point $(2,1)$. At $x = 2$ and $y = 1$, $\dfrac{dy}{dx} =$

A. 1

B. 2

C. 3

D. 4

E. 5

$$2y\frac{dy}{dx} + 8x = 3y + 3x\frac{dy}{dx} + 1$$

At $(2,1)$

$$2\frac{dy}{dx} + 16 = 3 + 6\frac{dy}{dx} + 1$$

$$12 = 4\frac{dy}{dx}$$

$$\frac{dy}{dx} = 3$$

NAME:______________________________ STUDENT NO:______________________________

9. *[4 marks]* Let $f(x) = 4xe^{-x^2}$. Then $f''(1) =$

A. 0

B. $4e^{-1}$

C. $-4e$

D. -8

E. $-8e^{-1}$

$f'(x) = 4e^{-x^2} - 8x^2e^{-x^2}$

$f''(x) = -8xe^{-x^2} - 16xe^{-x^2} + 16x^3e^{-x^2}$

$f''(1) = -8e^{-1} - 16e^{-1} + 16e^{-1}$

$= -8e^{-1}$

10. *[4 marks]* Let $f(x) = 2x^3 + 3x^2 + 6x + 5$. Then f is

A. increasing on $-\infty < x < -1.36$ and decreasing on $-1.36 < x < 0.36$

B. increasing everywhere

C. decreasing on $-\infty < x < -1.36$ and decreasing on $-1.36 < x < 0.36$.

D. decreasing everywhere.

E. increasing on $-1.36 < x < 0.36$ and decreasing on $0.36 < x < \infty$.

$f'(x) = 6x^2+6x+6 = 6(x^2+x+1)$ no real roots and $f(0) > 0$.

Hence $f' > 0$ everywhere

or $f'(x) = 6\left[(x+\frac{1}{2})^2 + \frac{3}{4}\right] \geq \frac{18}{4} > 0.$

f is increasing everywhere

NAME:______________________________ STUDENT NO:______________________________

PART B. Written Answer Questions

1. Given the input/output matrix for the two industries below, find the new outputs for each industry, if the final demand for each of the two industries changes to 32.

[15] [Any method for solving the resulting system of equations is acceptable, as long as you show the steps.]

	Ind A	Ind B	Final Demand
Ind A	5	20	25
Ind B	10	20	30
Other Production Factors	35	20	—

Let $\begin{pmatrix} x_A \\ x_B \end{pmatrix}$ = new output

Old Total production of Industry $A = 50$

" " " " " $B = 60$

A = Tech. Matrix $= \begin{pmatrix} \frac{5}{50} & \frac{20}{60} \\ \frac{10}{50} & \frac{20}{60} \end{pmatrix} = \begin{pmatrix} \frac{1}{10} & \frac{1}{3} \\ \frac{1}{5} & \frac{1}{3} \end{pmatrix}$

Leontieff Matrix $= I - A = \begin{pmatrix} \frac{9}{10} & -\frac{1}{3} \\ -\frac{1}{5} & \frac{2}{3} \end{pmatrix}$

$\det I-A = \frac{18}{30} - \frac{1}{15} = \frac{8}{15}$

$(I-A)^{-1} = \frac{15}{8}\begin{pmatrix} \frac{2}{3} & \frac{1}{3} \\ \frac{1}{5} & \frac{9}{10} \end{pmatrix} = \begin{pmatrix} \frac{5}{4} & \frac{5}{8} \\ \frac{3}{8} & \frac{27}{16} \end{pmatrix}$

$\begin{pmatrix} x_A \\ x_B \end{pmatrix} = \begin{pmatrix} \frac{5}{4} & \frac{5}{8} \\ \frac{3}{8} & \frac{27}{16} \end{pmatrix}\begin{pmatrix} 32 \\ 32 \end{pmatrix} = \begin{pmatrix} 40+20 \\ 12+54 \end{pmatrix} = \begin{pmatrix} 60 \\ 66 \end{pmatrix}$

$\boxed{x_A = 60 \qquad x_B = 66}$

Alternatively: Cramer

$x_A = \frac{15}{8}\begin{vmatrix} 32 & -\frac{1}{3} \\ 32 & \frac{2}{3} \end{vmatrix} = \frac{15}{8}\left(\frac{2}{3}\cdot 32 + \frac{1}{3}\cdot 32\right) = 60$

$x_B = \frac{15}{8}\begin{vmatrix} \frac{9}{10} & 32 \\ -\frac{1}{5} & 32 \end{vmatrix} = \frac{15}{8}\left(\frac{9}{10}\cdot 32 + \frac{2}{10}\cdot 32\right) = 66$

Other methods OK too.

NAME:______________________________ STUDENT NO:______________________________

2. Given $f(x) = \begin{cases} \dfrac{1-4^{-\frac{1}{x}}}{1+4^{-\frac{1}{x}}} & \text{if } x < 0 \\ x-1 & \text{if } 0 \le x \le 2 \\ \dfrac{x^2-6x+8}{x^2-5x+6} & \text{if } x > 2 \\ 5 & \text{if } x = 3 \end{cases}$

[10] a) Determine whether f is continuous for each of the following: (Justify your answer)

i) $x < 0$ cont

ii) $x = 0$ cont.

iii) $0 < x < 2$ cont.

iv) $x = 2$ not cont.

v) $x > 2$ cont. except for $x = 3$

Disconts of f can occur where $1+4^{-\frac{1}{x}} = 0$, but $4^{-\frac{1}{x}} > 0$; so not. or when $x^2-5x+6 = 0$ i.e. $(x-3)(x-2) = 0$: $x=2$ and $x=3$ and $x=0$. Need to check $x=0, x=2, x=3$. Cont. everywhere else.

$\lim_{x\to 0^-} f(x) = \lim_{x\to 0^-} \dfrac{4^{\frac{1}{x}}-1}{4^{\frac{1}{x}}+1} = -\frac{1}{1} = -1$ since $\lim_{x\to 0^-} 4^{\frac{1}{x}} = 0$

$f(0) = \lim_{x\to 0^+} f(x) = 0-1 = -1$ so $\boxed{\text{cont at } x=0}$

$f(2) = \lim_{x\to 2^-} f(x) = 1$ $\quad \lim_{x\to 2^+} f(x) = \lim_{x\to 2^+} \dfrac{(x-4)(x-2)}{(x-3)(x-2)} = \dfrac{-2}{-1} = 2$

$\boxed{\text{not cont at } x=2}$

not cont at $x=3$ since $f(x) = \dfrac{x-4}{x-3}$ $x \neq 2$ has no limit at $x=3$

[5] b) When is $f(x) \ge 0$ on the interval $x > 2$?

On $x > 2$, the only discont is at $x=3$ and the only zero is at $x=4$.

$f(x) = \begin{cases} \dfrac{x-4}{x-3} & x > 2 \quad x \neq 3 \\ 5 & x = 3 \end{cases}$

$(2,3)$ $f > 0$

$(3,4)$ $f < 0$

$(4,\infty)$ $f > 0$

At $x=3$, $f(3) = 5 > 0$.

At $x=4$, $f(4) = 0 \geq 0$

So $f(x) \geq 0$ on $(2,3] \cup [4,\infty)$

NAME:____________________ STUDENT NO:____________________

3. Suppose that a manufacturer's product has the demand curve $p(q) = e^{-0.01q}$ ($q =$ number of units bought; $p =$ unit price) and that, when he hires m employees, his output is $q = m^{\frac{1}{2}}$ (units produced).

[5] a) What is the marginal revenue when $q = 10$?

$$r = pq = qe^{-0.01q}$$

$$\frac{dr}{dq} = e^{-0.01q} - 0.01qe^{-0.01q} = (1-0.01q)e^{-0.01q}$$

$$\left.\frac{dr}{dq}\right|_{q=10} = \left(1-\frac{1}{10}\right)e^{-\frac{1}{10}} = \boxed{\frac{9}{10}e^{-\frac{1}{10}}} \approx \boxed{.814}$$

[5] b) What is the relative rate of change of revenue when $q = 10$?

$$\frac{1}{r}\frac{dr}{dq} = \frac{1-0.01q}{q}$$, So at $q=10$

$$\frac{1}{r}\frac{dr}{dq} = \frac{\frac{9}{10}}{10} = \boxed{\frac{9}{100} = .09}$$

[5] c) What is the marginal-revenue product $\frac{dr}{dm}$ when $m = 100$?

$$\frac{dr}{dm} = \frac{dr}{dq}\frac{dq}{dm} \qquad \frac{dq}{dm} = \frac{1}{2}m^{-\frac{1}{2}} = \frac{1}{2}\cdot\frac{1}{10}$$

at $m = 100$

and $q = 10$ when $m = 100$ so $\frac{dr}{dq} = \frac{9}{10}e^{-\frac{1}{10}}$

$$\left.\frac{dr}{dm}\right|_{m=100} = \frac{9}{10}e^{-\frac{1}{10}}\cdot\frac{1}{20} = \boxed{\frac{9e^{-\frac{1}{10}}}{200} \approx .0407}$$

or $\approx \boxed{.041}$

4.

[8] a) Find all the relative (local) maxima and minima of $f(x) = \dfrac{x^2 - 5x + 4}{x}$ on the interval $-10 < x < 10$.

Make sure to justify your answers.

$f(x) = x - 5 + \frac{4}{x}$

$f'(x) = 1 - \frac{4}{x^2}$

Crit pts at $x = \pm 2$ (Note that $x = 0$ is not in the domain of f.

	f'	f
$(-\infty, -2)$	+	inc
$(-2, 0)$	−	dec
$(0, 2)$	−	dec
$(2, \infty)$	+	inc

$\wedge$ $\boxed{x = -2 \text{ is a local max}}$

$\vee$ $\boxed{x = 2 \text{ is a local min}}$

2nd deriv test is acceptable

$f''(x) = \frac{8}{x^3}$

$f''(-2) = -1 < 0$ local max

$f''(2) = 1 > 0$ local min

4b) on next page. →

NAME:__________ STUDENT NO:__________

4.

[7] b) $y^2 e^{x-1} + x \ln y = 1$ defines y implicitly as a function of x near the point $(1,1)$. Find $\dfrac{d^2y}{dx^2}$ at this point. Please express your final answer as a decimal number.

$$2yy'e^{x-1} + y^2 e^{x-1} + \ln y + \frac{x}{y}y' = 0$$

at $(1,1)$: $2y' + 1 + y' = 0$ so $\boxed{y' = -\frac{1}{3}}$

$$2(y')^2 e^{x-1} + 2yy''e^{x-1} + 2yy'e^{x-1}$$

$$+ 2yy'e^{x-1} + y^2 e^{x-1}$$

$$+ \frac{1}{y}y'$$

$$+ \frac{1}{y}y' - \frac{x}{y^2}(y')^2 + \frac{x}{y}y''$$

$$\frac{2}{9} + 2y'' - \frac{2}{3} - \frac{2}{3} + 1 - \frac{1}{3} - \frac{1}{3} - \frac{1}{9} + y'' = 0$$

$$3y'' - \frac{8}{9} = 0$$

$$y'' = \boxed{\frac{8}{27} \approx .296}$$