Solved

Department of Mathematics
University of Toronto

MONDAY, DECEMBER 10, 2001, 9:00-11:00 AM
MAT 133Y TERM TEST #2

Calculus and Linear Algebra for Commerce
Duration: 2 hours

Aids Allowed: A non-graphing calculator, with empty memory, to be supplied by student.

Instructions: Fill in the information on this page, and make sure your test booklet contains 10 pages. In addition, you should have a **multiple-choice answer sheet**, on which you should fill in your name, number, tutorial time, tutorial room, and tutor's name.

This test consists of 10 multiple choice questions, and 4 written-answer questions.
For the **multiple choice questions** you can do your rough work in the test booklet, but you must record your answer by circling the appropriate letter **on the answer sheet** with your pencil. Each correct answer is worth 4 marks; a question left blank, or an incorrect answer, or two answers for the same question is worth 0. For the **written-answer questions**, present your solutions in the space provided. The value of each written-answer question is indicated beside it. ENCLOSE YOUR FINAL ANSWER IN A BOX AND WRITE IT IN INK.

TOTAL MARKS: 100

FAMILY NAME: ______________________

GIVEN NAME: ______________________

STUDENT NO: ______________________

SIGNATURE: ______________________

TUTORIAL TIME and ROOM: ______________________

REGCODE and TIMECODE: ______________________

T.A.'S NAME: ______________________

Regcode	Timecode	Room	**Regcode**	Timecode	Room
T0101A	M9A	SS2111	T0501C	W3C	NF 4
T0101B	M9B	SS2128	T0501D	W3D	CR 103
T0101C	M9C	SS2130	T0601A	R4A	UC 52
T0201A	M3A	LA 341	T0601B	R4B	UC 85
T0201B	M3B	UC 163	T0601C	R4C	UC 328
T0201C	M3C	SS2130	T0701A	F2A	UC 87
T0201D	M3D	LA 240	T0701B	F2B	WI 523
T0301A	T3A	SS2128	T0701C	F2C	SS1086
T0301B	T3B	SS1069	T0801A	F3A	SS2130
T0301C	T3C	SS2106	T0801B	F3B	SS2111
P0301D	T3D	VC 206	T5101A	R5A	UC 52
T0401A	W9A	SS1074	T5101B	R5B	UC 85
T0401B	W9B	SS2111	T5101C	R5C	UC 244
T0401C	W9C	LM 123	T5201A	R6A	UC 144
T0501A	W3A	TF 201	T5201B	R6B	UC 244
T0501B	W3B	TF 200			

FOR MARKER ONLY	
Multiple Choice	
B1	
B2	
B3	
B4	
TOTAL	

PART A. Multiple Choice

1. *[4 marks]*

$$\lim_{x\to 2} \frac{\sqrt{x+2}-2}{x-2} = \lim_{x\to 2} \frac{(x+2)-4}{(x-2)(\sqrt{x+2}+2)} = \lim_{x\to 2} \frac{1}{\sqrt{x+2}+2}$$

$$= \frac{1}{4}$$

A. $= -\frac{1}{4}$

B. $= 1$

C. $= 0$

D. $= \frac{1}{4}$

E. does not exist

2. *[4 marks]*
If an account earns money at the rate of 4% per year compounded continuously, how long does it take (to the nearest 0.1 year) for the amount in the account to triple?

A. 56.8 years

B. 18.7 years

C. 36.3 years

D. 75.0 years

E. 27.5 years

$$3P = Pe^{.04t}$$
$$3 = e^{.04t}$$
$$\ln 3 = .04t$$
$$t = \frac{\ln 3}{.04} \approx 27.47$$

3. *[4 marks]*
Let $f(x)$ be given for $0 \le x < 5$ only, by $f(x) = 3 - |x-2|$. Then f has

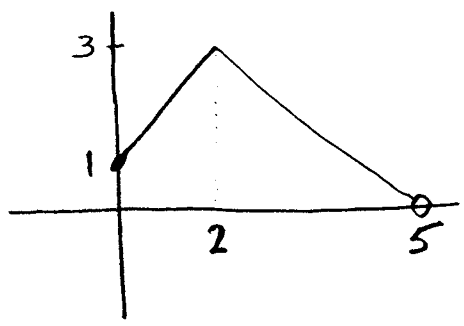
A. its absolute maximum at $x = 2$ and no absolute minimum.

B. its absolute maximum at $x = 3$ and absolute minimum at $x = 5$

C. its absolute maximum at $x = 2$ and absolute minimum at $x = 0$

D. neither an absolute maximum nor an absolute minimum

E. its absolute maximum at $x = 2$ and absolute minimum at $x = 5$



4. *[4 marks]*
If $f(x) = \sqrt{1+\sqrt{x}}$, then $f'(9) =$

A. $\frac{1}{24}$

B. $\frac{1}{12}$

C. $\frac{1}{4}$

D. $\frac{1}{3}$

E. $\frac{1}{2}$

$$f'(x) = \frac{1}{2\sqrt{1+\sqrt{x}}} \cdot \frac{1}{2\sqrt{x}}$$

$$f'(9) = \frac{1}{4\sqrt{1+3}\sqrt{9}} = \frac{1}{4 \cdot 2 \cdot 3} = \frac{1}{24}$$

5. *[4 marks]*
If the average cost per unit, $\bar{c}$, of producing q units of a product is given by

$$\bar{c} = \frac{4q}{q+2} + \frac{10,000}{q},$$

then the marginal cost when $q = 2$ is

A. -2499.5

B. 5002

C. 10,004

D. -5002

E. 3

$$c = \frac{4q^2}{q+2} + 10,000$$

$$\frac{dc}{dq} = 4\left[\frac{(q+2)2q - q^2}{(q+2)^2}\right]$$

$$= 4\left[\frac{q^2+4q}{(q+2)^2}\right]$$

$$\left.\frac{dc}{dq}\right|_{q=2} = 4\left[\frac{12}{16}\right] = 3$$

6. *[4 marks]*
Suppose $f'(x) = xf(x)$ for all values of x and $f(-2) = 3$. Find $f''(-2)$.

A. -1

B. -18

C. 1

D. 15

E. 12

$$f''(x) = f(x) + xf'(x)$$

$$f''(-2) = f(-2) - 2f'(-2)$$

but $f(-2) = 3$
and $f'(-2) = (-2)f(-2) = -6$

so $f''(-2) = 3 - 2(-6) = 15$

7. *[4 marks]*
If $f(u) = u^2$ and $h(x) = f(1 + g(x))$, $g'(1) = 1$, and $h'(1) = 1$, find $g(1)$.

A. -1

B. $-\frac{1}{2}$

C. 0

D. $\frac{1}{2}$

E. 1

$$h'(x) = f'(1+g(x))\,g'(x)$$
$$h'(1) = f'(1+g(1))\,g'(1)$$
$$1 = f'(1+g(1))$$

But $f(u) = u^2$ so $f'(u) = 2u$

$$1 = 2[1+g(1)]$$
$$\frac{1}{2} = 1 + g(1)$$
$$-\frac{1}{2} = g(1)$$

8. *[4 marks]*
If $y = \dfrac{\sqrt[3]{x^2+2x+7}}{\sqrt[5]{x^2+1}\,\sqrt[7]{3x+5}}$, then when $x = 0$, $\dfrac{y'}{y}$ is closest to

A. .0144

B. .0095

C. .0000

D. 1.913

E. 1.516

$$\ln y = \frac{1}{3}\ln(x^2+2x+7) - \frac{1}{5}\ln(x^2+1) - \frac{1}{7}\ln(3x+5)$$

$$\frac{1}{y}y' = \frac{2x+2}{3(x^2+2x+7)} - \frac{2x}{5(x^2+1)} - \frac{3}{7(3x+5)}$$

at $x=0$, $\dfrac{y'}{y} = \dfrac{2}{21} - \dfrac{3}{35} \approx .00952$

9. *[4 marks]*
If $y = (x+1)^{5x}$ when $x > 0$, then when $x = 1$, y' is closest to

A. 0

B. 22

C. 32

D. 90

E. 191

$\ln y = 5x \ln(x+1)$

$\frac{y'}{y} = 5 \ln(x+1) + \frac{5x}{x+1}$

when $x=1$, $y = 2^5 = 32$

so $y' = 32[5\ln 2 + \frac{5}{2}] \approx 190.90$

10. *[4 marks]*
Find $\dfrac{dy}{dx}$ if $\ln y = y \ln x$ defines y implicitly as a function x.

A. $\dfrac{y}{x - xy\ln x}$

B. $\dfrac{y^2}{x + xy\ln x}$

C. $\dfrac{y^2}{x - y\ln x}$

D. $\dfrac{y^2}{x - xy\ln x}$

E. $x^y \ln x$

$\frac{1}{y} y' = y' \ln x + \frac{y}{x}$

$y'(\frac{1}{y} - \ln x) = \frac{y}{x}$

$y'(1 - y\ln x) = \frac{y^2}{x}$

$y' = \frac{y^2}{x - xy\ln x}$

PART B. Written-Answer Questions

1. *[15 marks]*

Given the input-output matrix below for the two industries A and B, find the new outputs of these industries if the final demand for industries A and B changes to 200 and 400 respectively.

	Industry A	Industry B		Final Demand
Industry A	500	200	⋮	300
Industry B	400	200	⋮	400
Other Production Factors	100	600	⋮	–

Industry A produces 1000 units using 500 units from A $\left(a_{11} = \frac{500}{1000}\right)$ and 400 from B $\left(a_{21} = \frac{400}{1000}\right)$.

Industry B produces 1000 units, using 200 units from A $\left(a_{12} = \frac{200}{1000}\right)$ and 200 from B $\left(a_{22} = \frac{200}{1000}\right)$

$$A = \begin{pmatrix} \frac{1}{2} & \frac{1}{5} \\ \frac{2}{5} & \frac{1}{5} \end{pmatrix} \qquad I - A = \begin{pmatrix} \frac{1}{2} & -\frac{1}{5} \\ -\frac{2}{5} & \frac{4}{5} \end{pmatrix} \qquad \det(I-A) = \frac{2}{5} - \frac{2}{25} = \frac{8}{25}$$

$$(I-A)^{-1} = \frac{25}{8}\begin{pmatrix} \frac{4}{5} & \frac{1}{5} \\ \frac{2}{5} & \frac{1}{2} \end{pmatrix} = \begin{pmatrix} \frac{5}{2} & \frac{5}{8} \\ \frac{5}{4} & \frac{25}{16} \end{pmatrix}$$ can also be had by row-reduction.

$$(I-A)^{-1}\begin{pmatrix} 200 \\ 400 \end{pmatrix} = \begin{pmatrix} \frac{5}{2} & \frac{5}{8} \\ \frac{5}{4} & \frac{25}{16} \end{pmatrix}\begin{pmatrix} 200 \\ 400 \end{pmatrix} = \begin{pmatrix} 500 + 250 \\ 250 + 625 \end{pmatrix} = \begin{pmatrix} 750 \\ 875 \end{pmatrix}$$

$\boxed{\begin{array}{l} \text{750 units of A} \\ \text{875 units of B} \end{array}}$

Can also be gotten by solving the system

$$\left.\begin{aligned} x_A &= \tfrac{1}{2}x_A + \tfrac{1}{5}x_B + 200 \\ x_B &= \tfrac{2}{5}x_A + \tfrac{1}{5}x_B + 400 \end{aligned}\right\}$$ for the same answer

2. *[15 marks]*

Let

$$f(x) = \begin{cases} 5^{\frac{1}{x+1}} & \text{if } x < -1 \\ ax + b & \text{if } -1 \le x \le 2 \\ \dfrac{x^2-4}{x^2-x-2} & \text{if } 2 < x \end{cases}$$

where a and b are constants.

Are there values of a and b that make f continuous everywhere? If so, what are they? If not, why not?

$\lim_{x \to -1^-} f(x) = \lim_{x \to -1^-} 5^{\frac{1}{x+1}} = 0$ since $\frac{1}{x+1} \to -\infty$

$\lim_{x \to -1^+} f(x) = f(-1) = -a+b$ so $\boxed{b=a}$ will make f cont at $x=-1$

$\frac{x^2-4}{x^2-x-2} = \frac{(x-2)(x+2)}{(x-2)(x+1)} = \frac{x+2}{x+1}$ when $2 < x$

and this is cont. when $x > 2$

(Note that $x=-1$ doesn't come into it, since f is not defined by this fraction at $x=-1$.)

$\lim_{x \to 2^+} f(x) = \lim_{x \to 2^+} \frac{x+2}{x+1} = \frac{4}{3}$

$\lim_{x \to 2^-} f(x) = f(2) = 2a+b$

For continuity at $x=2$, $\boxed{2a+b = \frac{4}{3}}$

f is cont. everywhere if $a=b$ and $2a+b=\frac{4}{3}$

$\boxed{a = \frac{4}{9},\ b = \frac{4}{9}}$

3. *[6]* (a) Find **all** solutions of the inequality

$$\frac{x^3 - 4x^2}{x^2 - 4} \le 0.$$

$$\frac{x^2(x-4)}{(x-2)(x+2)}$$

$-\quad +\quad +\quad -\quad +$

$-2\quad 0\quad 2\quad 4$

$$\boxed{(-\infty, 2) \cup \{0\} \cup (2, 4]}$$

or $\boxed{x < 2 \quad \text{or} \quad x = 0 \quad \text{or} \quad 2 < x \le 4}$

[9] (b) Find **all** solutions of the inequality

$$\ln \frac{x+1}{x-1} > 1.$$

For ln to be defined, $\frac{x+1}{x-1} > 0$ is necessary,

so $x < -1$ or $x > 1$

For ln to be >1, $\frac{x+1}{x-1} > e$ is necessary

If $x < -1$, $x+1 < 0$, so $x - 1 < -2 < 0$. Multiply by $x-1$

$$x + 1 < (x-1)e = xe - e$$

$$e + 1 < xe - x = x(e-1)$$

$$1 < \frac{e+1}{e-1} < x$$

$x < -1$ and $x > 1$ is impossible!

If $x > 1$, $x - 1 > 0$

$$x + 1 > (x-1)e$$

$$e + 1 > xe - x = x(e-1)$$

$$\frac{e+1}{e-1} > x$$

Finally: $\boxed{1 < x < \frac{e+1}{e-1}}$

4. A manufacturer determines that m employees produce q units of a product per day, where $q = \dfrac{60m}{\sqrt{6+m}}$. The manufacturer currently has 10 employees.

[8] (a) If marginal revenue product is \$500.00, find the current marginal revenue. (Recall that marginal revenue product is $\frac{dr}{dm}$.)

[7] (b) If the number of units produced by these 10 employees could be increased by 1, it is known that in order to sell all the resulting production, unit price would have to be reduced by 0.1%. Find the unit price of the current level of production.

(a) $\dfrac{dr}{dm} = \dfrac{dr}{dq}\dfrac{dq}{dm}$

$$\frac{dq}{dm} = \frac{\sqrt{6+m}\cdot 60 - \dfrac{60m}{2\sqrt{6+m}}}{6+m}$$

$$\left.\frac{dq}{dm}\right|_{m=10} = \frac{60\left[4-\frac{5}{4}\right]}{16}$$

$$= 10.3125$$

$$500 = \frac{dr}{dq} \times 10.3125$$

$$\boxed{\frac{dr}{dq} \approx \$48.48}$$

b) $r = pq$

We are given $\dfrac{1}{p}\dfrac{dp}{dq} = -.001$

$\dfrac{dr}{dq} = q\dfrac{dp}{dq} + p$

and when $m=10$, $q = \dfrac{600}{4} = 150$

$\dfrac{dr}{dq} = pq\dfrac{1}{p}\dfrac{dp}{dq} + p$

$48.48 = p \times 150 \times (-.001) + p = -.15p + p = .85p$

$$p = \frac{48.48}{.85}$$

$$\boxed{p \approx \$57.04}$$