A

Department of Mathematics
University of Toronto

Tuesday, Nov. 1, 2016, 6:10-8:00 PM
MAT 133Y TERM TEST #1

Calculus and Linear Algebra for Commerce
Duration: 1 hour 50 minutes

Soln

Aids Allowed: A TI-30X IIS calculator, to be supplied by student. **No other calculator is permitted.**

Instructions: Fill in the information on this page, and make sure your test booklet contains 10 pages. In addition, you should have a **multiple-choice answer sheet**, on which you should fill in your name, number, tutorial time, tutorial room, and tutor's name.

This test consists of 10 multiple choice questions, and 4 written-answer questions.
For the **multiple choice questions** you can do your rough work in the test booklet, but you must record your answer by circling the appropriate letter **on the answer sheet** with your pencil. Each correct answer is worth 4 marks; a question left blank, or an incorrect answer, or two answers for the same question is worth 0. For the **written-answer questions**, present your solutions in the space provided. The value of each written-answer question is indicated beside it.

ENCLOSE YOUR FINAL ANSWER IN A BOX AND WRITE IT IN INK.

TOTAL MARKS: 100

FAMILY NAME: ____________________

GIVEN NAME: ____________________

STUDENT NO: ____________________

SIGNATURE: ____________________

TUTORIAL TIME and ROOM: ____________________

REGCODE and TIMECODE: ____________________

T.A.'S NAME: ____________________

Regcode	Timecode	Room	**Regcode**	Timecode	Room
T0101	M9A	HA316	T0502	W3B	PB255
T0102	M9B	HA401	T0503	W3C	UC87
T0103	M9C	HA410	T0601	R4A	BA3012
T0104	M9D	LM157	T0602	R4B	PB255
T0201	M3A	MS4171	T0603	R4C	MP134
T0202	M3B	PB255	T0604	R4D	SS2110
T0203	M3C	ES B149	T0701	F2A	MP134
T0204	M3D	AB107	T0702	F2B	SS2110
T0301	T3A	PB255	T0703	F2C	SS1073
T0302	T3B	UC52	T0801	F3A	MP134
T0303	T3C	AP120	T0802	F3B	SS2110
T0304	T3D	BA2175	T0803	F3C	SS1073
T0401	W9A	AB114	T5101	M5A	AB114
T0402	W9B	BA2159	T5102	M5B	AP120
T0403	W9C	BA2175	T5103	M5C	BA3116
T0501	W3A	BA2155	T5104	M5D	HA316

FOR MARKER ONLY	
Multiple Choice	
B1	
B2	
B3	
B4	
TOTAL	

NAME: ______________________ STUDENT NO: ______________

PART A. Multiple Choice

1. *[4 marks]*

You bought \$70 worth of stock A on Jan 1, 2013, and the stock was worth \$100 on Jan 1, 2015. The effective annual rate of this investment over that period of time is closest to

A. 30%

B. 21.4%

C. 19.5%

D. 15%

E. 10.7%

0 1 2

$70 \longrightarrow 100$

Let r_e = effective annual rate

$70(1+r_e)^2 = 100$

$1+r_e = \left(\frac{100}{70}\right)^{\frac{1}{2}} = 1.1952\ldots$

$r_e = .1952\ldots$ $\boxed{C}$

2. *[4 marks]*

Dave receives X dollars at the end of the first year and 3 times as much at the end of the third year. All this income together is valued at \$5000 at the end of the fourth year. If interest is 6% per year compounded semi-annually, then, to the nearest dollar, X is

A. \$857

B. \$1,142

C. \$1,288

D. \$713

E. \$1,023

0 2 6 8

X $3X$ $\longrightarrow 5000$

$r = .03$

$X(1.03)^6 + 3X(1.03)^2 = 5000$

$X = \frac{5000}{(1.03)^6 + 3(1.03)^2} = 1142.40$

$\boxed{B}$

NAME: ____________________ STUDENT NO: ____________

3. *[4 marks]*

A debt of $400 is due at the end of 3 years and a debt of $600 is due at the end of 5 years. Money is worth 1% per year compounded quarterly. The amount of a single payment at the end of 4 years that pays off both debts is closest to

A. $998.06

B. $1,000.00

C. $999.50

D. $1,010.00

E. $992.83

0 12 16 20 $r = .0025$

400 X 600

Let X be the amount of the payment.

$400(1.0025)^4 + 600(1.0025)^{-4} = X$

$X = 998.05$ $\boxed{A}$

4. *[4 marks]*

A $100,000 loan at 3% per year compounded monthly is amortized over 20 years. The monthly payments are nearest to

A. $554.60

B. $552.92

C. $304.60

D. $3,002.49

E. $3,000.07

$i = \frac{.03}{12}$ R = monthly payment

$100{,}000 = R\, a_{\overline{240|}.0025}$

$R = \frac{100{,}000}{a_{\overline{240|}.0025}}$

$= \frac{100{,}000 \times .0025}{1 - (1.0025)^{-240}}$

$R \approx 554.60$ $\boxed{A}$

NAME: ______________________ STUDENT NO: ______________

5. *[4 marks]*

Erica wants to retire at age 65. She would like to invest in a perpetuity and receive \$50,000 per year at the end of each year after she retires. If the effective annual interest rate is 3.5% per year, how much money must she invest at age 65, to the nearest dollar?

First payment is 1 yr after retirement so this just a perpetuity

$R = r A$ where R is the annual payment $= 50{,}000$, r = rate per period $= .035$ and A = amount to invest

$A = \frac{R}{r}$

$= \frac{50{,}000}{.035} = 1{,}428{,}571$ $\boxed{B}$

A. \$964, 252

B. \$1, 428, 571

C. \$349, 211

D. \$830, 419

E. \$1, 205, 033

6. *[4 marks]*

Six monthly deposits of \$500 are made into an account that earns 6% per year compounded monthly. One month after the last deposit, the account balance is closest to

Could relabel: −1 0 1 … 5 6

0 1 2 … 6 7

500 500 … 500

$r = \frac{.06}{12}$

S = balance

$S = 500\, s_{\overline{6}|.005}(1.005)$

$= 500\left[\frac{(1.005)^6 - 1}{.005}\right] \times 1.005 \approx 3052.94$ $\boxed{D}$

or: 0 1 2 … 6 7

500 500 … 500 (ghost 500)

$S = 500\, s_{\overline{7}|.005} - 500$

$= 3052.94$ as before

A. \$2, 948.19

B. \$2, 962.93

C. \$3, 037.75

D. \$3, 052.94

E. \$3, 063.63

NAME: ____________________ STUDENT NO: ______________

7. *[4 marks]*

Let $A = \begin{bmatrix} 3 & -1 & 1 \\ 0 & 2 & 1 \end{bmatrix}$, $B = \begin{bmatrix} 4 & 1 \\ 0 & 2 \end{bmatrix}$ and $C = \begin{bmatrix} 1 & 4 & 2 \\ 3 & 1 & 5 \end{bmatrix}$. The entry in the first row and the second column of the matrix BCA^T is

A. 47

B. 33

C. 27

D. 26

E. the product BCA^T is not defined

$$BCA^T = \begin{pmatrix} 4 & 1 \\ 0 & 2 \end{pmatrix}\begin{pmatrix} 1 & 4 & 2 \\ 3 & 1 & 5 \end{pmatrix}\begin{pmatrix} 3 & 0 \\ -1 & 2 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} 4 & 1 \\ 0 & 2 \end{pmatrix}\begin{pmatrix} 1 & 10 \\ 13 & 7 \end{pmatrix} = \begin{pmatrix} \cdot & 47 \\ \cdot & \cdot \end{pmatrix}$$

Actually:

$$BCA^T = \begin{pmatrix} 17 & 47 \\ 26 & 14 \end{pmatrix}$$

$\boxed{A}$

8. *[4 marks]*

Let $A = \begin{bmatrix} 0 & 1 \\ 0 & 2 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 1 \\ 3 & 4 \end{bmatrix}$ and $C = \begin{bmatrix} 2 & 5 \\ 3 & 4 \end{bmatrix}$. The matrix $AB - AC^T$ is equal to

A. $\begin{bmatrix} -2 & 0 \\ -4 & 0 \end{bmatrix}$

B. $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

C. $\begin{bmatrix} -5 & -4 \\ -3 & 1 \end{bmatrix}$

D. $\begin{bmatrix} 1 & 1 \\ -4 & 0 \end{bmatrix}$

E. $\begin{bmatrix} -2 & 0 \\ 2 & 2 \end{bmatrix}$

Easiest way: $AB - AC^T = A(B - C^T)$

$$B - C^T = \begin{pmatrix} 1 & 1 \\ 3 & 4 \end{pmatrix} - \begin{pmatrix} 2 & 3 \\ 5 & 4 \end{pmatrix} = \begin{pmatrix} -1 & -2 \\ -2 & 0 \end{pmatrix}$$

$$A(B - C^T) = \begin{pmatrix} 0 & 1 \\ 0 & 2 \end{pmatrix}\begin{pmatrix} -1 & -2 \\ -2 & 0 \end{pmatrix} = \begin{pmatrix} -2 & 0 \\ -4 & 0 \end{pmatrix}$$

$\boxed{A}$

A

NAME: ____________________ STUDENT NO: ____________

9. *[4 marks]*

The system of linear equations

$$\begin{cases} x + 2y + 3z = 1 \\ 2x + 5y + 3z = 4 \\ x + 8z = 2 \end{cases}$$

has

$$\left(\begin{array}{ccc|c} 1 & 2 & 3 & 1 \\ 2 & 5 & 3 & 4 \\ 1 & 0 & 8 & 2 \end{array}\right) \xrightarrow[R_3 \to -R_1 + R_3]{R_2 \to -2R_1 + R_2} \left(\begin{array}{ccc|c} 1 & 2 & 3 & 1 \\ 0 & 1 & -3 & 2 \\ 0 & -2 & 5 & 1 \end{array}\right)$$

$$\xrightarrow{R_3 \to 2R_2 + R_3} \left(\begin{array}{ccc|c} 1 & 2 & 3 & 1 \\ 0 & 1 & -3 & 2 \\ 0 & 0 & -1 & 5 \end{array}\right) \xrightarrow{R_3 \to -R_3} \left(\begin{array}{ccc|c} 1 & 2 & 3 & 1 \\ 0 & 1 & -3 & 2 \\ 0 & 0 & 1 & -5 \end{array}\right)$$

A. no solution

B. a unique solution with $z = -5$

C. a unique solution with $z = 1$

D. a one-parameter family of solutions

E. a two-parameter family of solutions

This already says:
Unique solution with $z = -5$.
$\boxed{B}$

10. *[4 marks]*

If X and Y are 2×2 matrices, only **one** of the following statements is always true. Which one?

A. If $XY = 0$ then at least one of X or Y is 0. False: $\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ -1 & -1 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$

B. $XY = YX$. False: $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} c & d \\ a & b \end{pmatrix}$ but $\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} b & a \\ d & c \end{pmatrix}$

C. $(X + Y)^2 = X^2 + 2XY + Y^2$. False: $(X+Y)(X+Y) = X^2 + XY + YX + Y^2 = X^2 + 2XY + Y^2$ only if $XY = YX$ (see B.)

D. If $X \neq 0$, then X has an inverse X^{-1}. False: $\begin{pmatrix} a & b \\ 0 & 0 \end{pmatrix}$ has no inverse.

E. If X and Y have inverses X^{-1} and Y^{-1}, then $(XY)^{-1} = Y^{-1}X^{-1}$.

Must be $\boxed{E}$ and in fact:

$$(Y^{-1}X^{-1})(XY) = Y^{-1}(X^{-1}X)Y = Y^{-1}IY$$
$$= Y^{-1}Y = I \text{ as required.}$$

Similarly $(XY)(Y^{-1}X^{-1}) = I$.

NAME: ______________________ STUDENT NO: ______________

PART B. Written-Answer Questions

1. *[16 marks]*

A $450,000 loan is amortized over 20 years with semi-annual payments at an interest rate of 4% per year compounded semi-annually.

[5] (a) What are the semi-annual payments? R $\quad i = .02$

$$450{,}000 = R\, a_{\overline{40|}.02}$$

$$R = \frac{450{,}000}{a_{\overline{40|}.02}} = \frac{450{,}000 \times .02}{1-(1.02)^{-40}} = \boxed{\$16{,}450.09}$$

[5] (b) How much are the interest and principal in the last payment?

At the beginning of the last period, only the last payment remains, so the principal outstanding is

$$R(1+i)^{-1} = 16{,}450.09\,(1.02)^{-1} = 16{,}127.54$$

and the interest for the period is $iR(1+i)^{-1} = \boxed{\$322.55} =$ Interest in last paymt

So the principal in the last payment

$$= 16{,}450.09 - 322.55 = \boxed{\$16{,}127.54}$$

[6] (c) After 10 years (so with 10 years remaining) the lender and borrower agree to re-amortize the last 10 years of payments at the new interest rate of 5% per year compounded annually with annual payments (the first one at the end of the 11th year). How big are these annual payments?

After 10 years, the principal outstanding is

$$R\, a_{\overline{20|}.02} = 16{,}450.09 \frac{\left[1.-(1.02)^{-20}\right]}{.02}$$

$$P.O = 268{,}982.55$$

Re-amortized over 10 annual payments at 5% p.a.

$$268{,}982.55 = T\, a_{\overline{10|}.05} \qquad (T \text{ is new annual paymt})$$

$$T = \frac{268{,}982.55 \times .05}{1-(1.05)^{-10}}$$

$$= \boxed{\$34{,}834.47}$$

NAME: ______________________ STUDENT NO: ______________

2. *[16 marks]* The rate of interest (i.e. yield rate) is 3.25% per year compounded semi-annually.

[10] (a) Which of the following has more value at the present time?

(i) A mortgage with 240 monthly payments of \$600 remaining (the first payment in 1 month's time).

$$(1+i)^{12} = \left(1+\frac{.0325}{2}\right)^2$$

$$A = 600\, a_{\overline{240}|i} = \frac{600\left[1-(1+i)^{-240}\right]}{i}$$

$$= \frac{600\left[1-\left(1+\frac{.0325}{2}\right)^{-40}\right]}{\left(1+\frac{.0325}{2}\right)^{\frac{1}{6}}-1}$$

$$\boxed{A = \$105,990.00}$$

or (ii) A \$100,000 bond with an annual coupon rate of 3.5% and 43 semi-annual interest payments remaining (the first one in 6 months time).

$V = 100,000 \quad rV = \frac{.035}{2} \times 100,000 = \$1,750$ the coupon

$$P = 100,000\left(1+\frac{.0325}{2}\right)^{-43} + 1,750\, a_{\overline{43}|\frac{.0325}{2}}$$

$$\boxed{P = \$103,846.10}$$

The mortgage has more value.

[6] (b) If the yield rate remains the same, what would the annual coupon rate of the bond have to be for the mortgage and the bond to have the same value at the present time?

Let x be the new annual coupon rate (necessarily more than .0325.

The new semi-annual coupon is $\frac{x}{2} \times 100,000$ and we are trying to move the price to

$$105,990 = 100,000\left(1+\frac{.0325}{2}\right)^{-43} + 100,000\frac{x}{2}\, a_{\overline{43}|\frac{.0325}{2}}$$

$$\frac{105,990 - 100,000(1.01625)^{-43}}{50,000\, a_{\overline{43}|.01625}} = x$$

$x = .03639355\ldots$

Annual coupon rate must be raised to $\approx \boxed{3.64\%}$

A

NAME: ______________________ STUDENT NO: ______________

3. *[13 marks]*

An animal shelter has three times as many dogs as hamsters. The number of cats is equal to the number of dogs plus six times the number of hamsters. The number of cats plus seven times the number of ferrets is equal to two hundred and fifty. There are 50 more dogs than ferrets. Express these facts as a linear system and use row reduction to solve. How many of each kind of animal are in the shelter?
[6 marks for setting up the system and 7 marks for solving it using row reduction: no marks will be given for any other method.]

Let D = # of dogs, H = # of hamsters, C = # cats, F = # ferrets

$D = 3H$
$C = D + 6H$
$C + 7F = 250$
$D - F = 50$

$$\left.\begin{aligned} D - 3H &= 0 \\ D + 6H - C &= 0 \\ C + 7F &= 250 \\ D - F &= 50 \end{aligned}\right\}$$

the system of equations

$$\begin{array}{cccc} D & H & C & F \end{array}$$

$$\left(\begin{array}{cccc|c} 1 & -3 & 0 & 0 & 0 \\ 1 & 6 & -1 & 0 & 0 \\ 0 & 0 & 1 & 7 & 250 \\ 1 & 0 & 0 & -1 & 50 \end{array}\right) \xrightarrow[R_4 \to -R_1 + R_4]{R_2 \to -R_1 + R_2} \left(\begin{array}{cccc|c} 1 & -3 & 0 & 0 & 0 \\ 0 & 9 & -1 & 0 & 0 \\ 0 & 0 & 1 & 7 & 250 \\ 0 & 3 & 0 & -1 & 50 \end{array}\right)$$

$$\xrightarrow[R_4 \to -3R_2 + R_4]{R_2 \to \frac{1}{9}R_2} \left(\begin{array}{cccc|c} 1 & -3 & 0 & 0 & 0 \\ 0 & 1 & -\frac{1}{9} & 0 & 0 \\ 0 & 0 & 1 & 7 & 250 \\ 0 & 0 & \frac{1}{3} & -1 & 50 \end{array}\right) \xrightarrow[R_4 \to -\frac{1}{3}R_3 + R_4]{} \left(\begin{array}{cccc|c} 1 & -3 & 0 & 0 & 0 \\ 0 & 1 & -\frac{1}{9} & 0 & 0 \\ 0 & 0 & 1 & 7 & 250 \\ 0 & 0 & 0 & -\frac{10}{3} & -\frac{100}{3} \end{array}\right)$$

$$\begin{array}{cccc} D & H & C & F \end{array}$$

$$\xrightarrow{R_4 \to -\frac{3}{10}R_4} \left(\begin{array}{cccc|c} 1 & -3 & 0 & 0 & 0 \\ 0 & 1 & -\frac{1}{9} & 0 & 0 \\ 0 & 0 & 1 & 7 & 250 \\ 0 & 0 & 0 & 1 & 10 \end{array}\right)$$

so $F = 10$

$C = 250 - 7F = 250 - 70 = 180$

$H = \frac{1}{9}C = \frac{1}{9} \times 180 = 20$

and $D = 3H = 3 \times 20 = 60$

$\boxed{\text{60 dogs, 20 hamsters, 180 cats, 10 ferrets}}$

Note: It goes more easily if you put the hamsters last instead of second (as in the soln above) but it comes out the same.

A

NAME: ____________________ STUDENT NO: ________________

4. *[15 marks]*

Consider the matrix $A = \begin{bmatrix} -1 & 2 & 0 \\ 2 & -1 & -3 \\ -1 & 2 & 1 \end{bmatrix}$

[10] (a) Find the inverse of A by row reducing the matrix $[A|I_3]$.

$$\left(\begin{array}{ccc|ccc} -1 & 2 & 0 & 1 & 0 & 0 \\ 2 & -1 & -3 & 0 & 1 & 0 \\ -1 & 2 & 1 & 0 & 0 & 1 \end{array}\right) \xrightarrow[\substack{R_2 \to -2R_1 + R_2 \\ R_3 \to R_1 + R_3}]{R_1 \to -R_1} \left(\begin{array}{ccc|ccc} 1 & -2 & 0 & -1 & 0 & 0 \\ 0 & 3 & -3 & 2 & 1 & 0 \\ 0 & 0 & 1 & -1 & 0 & 1 \end{array}\right)$$

$$\xrightarrow{R_2 \to \frac{1}{3} R_2} \left(\begin{array}{ccc|ccc} 1 & -2 & 0 & -1 & 0 & 0 \\ 0 & 1 & -1 & \frac{2}{3} & \frac{1}{3} & 0 \\ 0 & 0 & 1 & -1 & 0 & 1 \end{array}\right) \xrightarrow[R_1 \to 2R_2 + R_1]{R_2 \to R_3 + R_2} \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & -\frac{5}{3} & \frac{2}{3} & 2 \\ 0 & 1 & 0 & -\frac{1}{3} & \frac{1}{3} & 1 \\ 0 & 0 & 1 & -1 & 0 & 1 \end{array}\right)$$

$$\boxed{A^{-1} = \begin{pmatrix} -\frac{5}{3} & \frac{2}{3} & 2 \\ -\frac{1}{3} & \frac{1}{3} & 1 \\ -1 & 0 & 1 \end{pmatrix} = \frac{1}{3} \begin{pmatrix} -5 & 2 & 6 \\ -1 & 1 & 3 \\ -3 & 0 & 3 \end{pmatrix}}$$

Check:

$$\begin{pmatrix} -1 & 2 & 0 \\ 2 & -1 & -3 \\ -1 & 2 & 1 \end{pmatrix} = \frac{1}{3} \begin{pmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{pmatrix} \checkmark$$

[5] (b) Solve the system of equations

$$\begin{cases} -x + 2y = 3 \\ 2x - y - 3z = 6 \\ -x + 2y + z = 1 \end{cases}$$

The system is $A \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 3 \\ 6 \\ 1 \end{pmatrix}$ where $A = \begin{pmatrix} -1 & 2 & 0 \\ 2 & -1 & -3 \\ -1 & 2 & 1 \end{pmatrix}$

so $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = A^{-1} \begin{pmatrix} 3 \\ 6 \\ 1 \end{pmatrix} = \frac{1}{3} \begin{pmatrix} -5 & 2 & 6 \\ -1 & 1 & 3 \\ -3 & 0 & 3 \end{pmatrix} \begin{pmatrix} 3 \\ 6 \\ 1 \end{pmatrix}$

$$= \frac{1}{3} \begin{pmatrix} -15 + 12 + 6 \\ -3 + 6 + 3 \\ -9 + 0 + 3 \end{pmatrix} = \frac{1}{3} \begin{pmatrix} 3 \\ 6 \\ -6 \end{pmatrix} = \boxed{\begin{pmatrix} 1 \\ 2 \\ -2 \end{pmatrix}}$$

so $\boxed{\begin{array}{l} x = 1 \\ y = 2 \\ z = -2 \end{array}}$