A

Department of Mathematics
University of Toronto
Tuesday, October 27, 2015, 6:10-8:00 PM
MAT 133Y TERM TEST #1
Calculus and Linear Algebra for Commerce
Duration: 1 hour 50 minutes

Soln.

Aids Allowed: A non-graphing calculator, with empty memory, to be supplied by student.

Instructions: Fill in the information on this page, and make sure your test booklet contains 10 pages. In addition, you should have a **multiple-choice answer sheet**, on which you should fill in your name, number, tutorial time, tutorial room, and tutor's name.

This test consists of 10 multiple choice questions, and 4 written-answer questions.
For the **multiple choice questions** you can do your rough work in the test booklet, but you must record your answer by circling the appropriate letter **on the answer sheet** with your pencil. Each correct answer is worth 4 marks; a question left blank, or an incorrect answer, or two answers for the same question is worth 0. For the **written-answer questions**, present your solutions in the space provided. The value of each written-answer question is indicated beside it.

ENCLOSE YOUR FINAL ANSWER IN A BOX AND WRITE IT IN INK.

TOTAL MARKS: 100

FAMILY NAME: _____

GIVEN NAME: _____

STUDENT NO: _____

SIGNATURE: _____

TUTORIAL TIME and ROOM: _____

REGCODE and TIMECODE: _____

T.A.'S NAME: _____

Regcode	Timecode	Room	**Regcode**	Timecode	Room
T0101A	M9A	BA2135	T0501B	W3B	SS2105
T0101B	M9B	BA2165	T0501C	W3C	UC52
T0101C	M9C	BA1240	T0601A	R4A	BL112
T0101D	M9D	BA2139	T0601B	R4B	BL114
T0201A	M3A	BA B024	T0601C	R4C	SS562
T0201B	M3B	RW142	T0601D	R4D	UC114
T0201C	M3C	WO25	T0701A	F2A	AP120
T0201D	M3D	WW119	T0701B	F2B	BF323
T0301A	T3A	RW142	T0701C	F2C	LM155
T0301B	T3B	HA316	T0801A	F3A	AP120
T0301C	T3C	SS1086	T0801B	F3B	BF323
T0301D	T3D	SS2111	T0801C	F3C	LM155
T0401A	W9A	BA2195	T5101A	M5A	AP120
T0401B	W9B	AP120	T5101B	M5B	BA2175
T0401C	W9C	LM155	T5101C	M5C	BA2185
T0401D	W9D	BA2159	T5101D	M5D	SS2111
T0501A	W3A	HA316			

FOR MARKER ONLY	
Multiple Choice	
B1	
B2	
B3	
B4	
TOTAL	

NAME: _____ STUDENT NO: _____

PART A. Multiple Choice

1. *[4 marks]*

A nominal rate of 4% compounded quarterly is equivalent to what nominal rate compounded semi-annually?

Let r = nominal rate compounded semi-annually

$\left(1+\frac{r}{2}\right)^2 = (1.01)^4$

$1+\frac{r}{2} = (1.01)^2 = 1.0201$

$\frac{r}{2} = .0201$

$r = .0402$

4.02% (B)

A. 4.01%

B. 4.02%

C. 4.03%

D. 4.04%

E. 4.05%

2. *[4 marks]*

After 4 years sitting in an account with interest compounded semi-annually, an initial investment of \$2000 grows into \$2844.20. The nominal annual rate is closest to

Let r = nominal rate of interest

$2844.20 = 2000\left(1+\frac{r}{2}\right)^8$

$2\left[\left(\frac{2844.20}{2000}\right)^{\frac{1}{8}} - 1\right] = r$

$.0899999 = r$ (D)

A. 4.5%

B. 4.6%

C. 8%

D. 9%

E. 9.2%

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3. *[4 marks]*

A person borrows \$5000 today at 6% compounded semi-annually. If he pays \$1000 one year from now and \$2000 two years from now, then how much must he pay 3 years from now in order to completely pay off the loan?

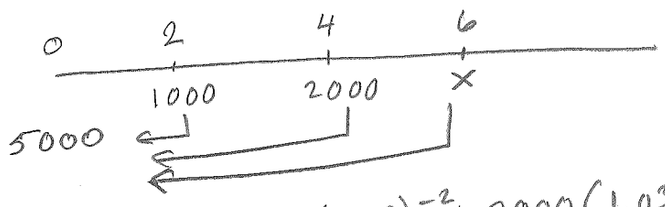
A. \$2000

B. \$2572.60

C. \$2722.95

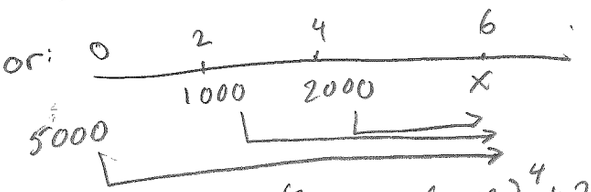
D. \$2121.80

E. \$2120



$$5000 = 1000(1.03)^{-2} + 2000(1.03)^{-4} + x(1.03)^{-6}$$

or:



$$5000(1.03)^6 = 1000(1.03)^4 + 2000(1.03)^2 + X$$

In any case $X = \#2722.95$ (C)

4. *[4 marks]*

Every day for one year (of 365 days), a student deposits one dollar into an account earning 11% per year compounded daily. At the beginning the account holds \$0. How much is in the account after the last deposit?

A. \$377.98

B. \$385.77

C. \$387.74

D. \$405.15

E. \$956.30

$$S = 1 \times s_{\overline{365}|\frac{.11}{365}}$$

$$= \frac{\left(1 + \frac{.11}{365}\right)^{365} - 1}{\frac{.11}{365}}$$

$$= \#385.77 \quad (B)$$

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5. *[4 marks]*

Mr. Khaghani wants to make his retirement easy by setting up a perpetuity that gives him \$1500 per month for groceries and weekly outings to a local jazz club. How much money must he put in to the fund if his money earns a nominal interest rate of 16% per year compounded monthly?

$r = \frac{.16}{12}$ $\quad R = rA$

$1500 = \frac{.16}{12} A$

$A = \frac{12 \times 1500}{.16} = \$112,500$

(E)

A. \$140,215

B. \$131,200

C. \$122,750

D. \$115,000

E. \$112,500

6. *[4 marks]*

Maria takes out a \$500,000 mortgage amortized over 30 years at an interest rate of 6% per year compounded semi-annually with monthly payments at the end of each month. The monthly payments are closest to:

$(1.03)^2 = (1+i)^{12}$ $\quad (1.03)^{-60} = (1+i)^{-360}$

$500,000 = R\, a_{\overline{360}|i}$

$R = \frac{500,000}{a_{\overline{360}|i}}$

$= \frac{500,000\, i}{1-(1+i)^{-360}}$

$= 500,000 \frac{[(1.03)^{\frac{1}{6}} - 1]}{1-(1.03)^{-60}}$

$R = \$2974.11$ (A)

A. \$2974

B. \$2998

C. \$2959

D. \$1396

E. \$1986

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7. *[4 marks]*

Let $A = \begin{bmatrix} 3 & 2 \\ 0 & -5 \end{bmatrix}$ and $B = \begin{bmatrix} -1 & 0 \\ 7 & 11 \end{bmatrix}$.

Let $A + 2B = C$ where the entries of C are C_{ij}. Then $C_{21} =$

A. 1

B. 2

C. 14

D. 16

E. 17

$C_{21} = A_{21} + 2B_{21}$
$= 0 + 2 \cdot 7 = 14$ (C)

or $A + 2B = \begin{pmatrix} 3 & 2 \\ 0 & -5 \end{pmatrix} + \begin{pmatrix} -2 & 0 \\ 14 & 22 \end{pmatrix}$

$= \begin{pmatrix} 1 & 2 \\ 14 & 17 \end{pmatrix} = C$

$C_{21} = 14$ as before

8. *[4 marks]*

Let A be a 7×3 matrix, B be a 5×7 matrix, and C be a 5×3 matrix. Then only one of the following is defined. Which one?

A. $A + B - C$ A, B and C are not the same shape: not defined

B. $CB - A$ $5\overset{3}{C}\,5\overset{7}{B}$ CB not defined

C. $CA + B$ $5\overset{3}{C}\,7\overset{3}{A}$ CA not defined

D. $AB + C$ $7\overset{3}{A}\,5\overset{7}{B}$ AB not defined

E. $BA - C$ $5\overset{7}{B}\,7\overset{3}{A}$ result is 5×3 and C is indeed 5×3 so defined (E)

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9. *[4 marks]*

If $A = \begin{bmatrix} 2 & 1 & 0 \\ -1 & 0 & 1 \\ 0 & -2 & -1 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 0 & 1 \\ 0 & -1 & 2 \end{bmatrix}$, $I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$,

$C = IAB^T$, and the entries of C are C_{ij}, then $C_{22} =$

A. -2

B. -1

C. 1

D. 2

E. 0

$C = AB^T$ since multiply I doesn't change anything.

$= \begin{pmatrix} 2 & 1 & 0 \\ -1 & 0 & 1 \\ 0 & -2 & -1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -1 \\ 1 & 2 \end{pmatrix}$

To get C_{22}, take the 2nd row of A and the 2nd column of B^T

$-1 \quad 0 \quad 1$ and $0 \quad -1 \quad 2$

$-1 \cdot 0 + 0 \cdot (-1) + 1 \cdot 2 = 2 = C_{22}$ (D)

10. *[4 marks]*

The system

$$\begin{aligned} x - y + z &= 4 \\ 2x + 3z &= 3 \\ 2y + z &= 5 \end{aligned}$$

has

A. a unique solution with $z = 5$

B. a unique solution with $z = 10$

C. a unique solution with $y = 1$

D. infinitely many solutions

E. no solution

$\left(\begin{array}{ccc|c} 1 & -1 & 1 & 4 \\ 2 & 0 & 3 & 3 \\ 0 & 2 & 1 & 5 \end{array}\right)$

$R_2 \to -2R_1 + R_2$ $\left(\begin{array}{ccc|c} 1 & -1 & 1 & 4 \\ 0 & 2 & 1 & -5 \\ 0 & 2 & 1 & 5 \end{array}\right)$

$R_3 \to -R_2 + R_3$ $\left(\begin{array}{ccc|c} 1 & -1 & 1 & 4 \\ 0 & 2 & 1 & -5 \\ 0 & 0 & 0 & 10 \end{array}\right)$ ←

No solution (E)

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PART B. Written-Answer Questions

1. *[15 marks]*

John takes out a loan from a wealthy friend for \$250,000 and promises to pay it back over 20 years. His friend insists that he pay back the loan in monthly installments, beginning at the end of the first month, and is charging 5% per year compounded monthly.

[7] (a) What is the amount John must pay his friend each month?

$$250{,}000 = R\, a_{\overline{240}|\frac{.05}{12}}$$

$$R = \frac{250{,}000}{a_{\overline{240}|\frac{.05}{12}}} = \frac{250{,}000 \times \frac{.05}{12}}{1-\left(1+\frac{.05}{12}\right)^{-240}}$$

$$\boxed{R = \$1649.89}$$

[8] (b) After 10 years pass, John and his friend decide to renegotiate the loan. They agree to the following new terms in paying off the balance: the interest rate will be 5.5% per year compounded semi-annually and the payments will be every 6 months, with the first new payment at the end of the first 6 months. How much must John pay every 6 months?

Principal outstanding $= R\, a_{\overline{120}|\frac{.05}{12}}$ because 120 payments remain.

$$= 1649.89 \frac{\left[1-\left(1+\frac{.05}{12}\right)^{-120}\right]}{\frac{.05}{12}}$$

$$= \$155{,}553.86$$

If we call the new payments T

$$155{,}553.86 = T\, a_{\overline{20}|\frac{.055}{2}}$$

$$T = \frac{155{,}553.86 \times \frac{.055}{2}}{1-\left(1+\frac{.055}{2}\right)^{-20}}$$

$$\boxed{T = \$10{,}215.49}$$

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2. *[15 marks]*

Lina buys a bond with 9 semi-annual coupons remaining at an annual coupon rate of 4.6% at a price of \$98.41 per \$100 of face value. Find the annual yield to maturity of the bond.
(Your answer will be sufficiently accurate if the price comes out within \$0.50 of the actual price.)

$98.41 = 100(1+i)^{-9} + 2.30\, a_{\overline{9}|i}$

Since the price is less than \$100, the semi-annual yield must be more than .023

Let's try .03

$100(1.03)^{-9} + 2.30\, a_{\overline{9}|.03} = 94.55$ much too low a price

\$100 was closer

So .03 is much too high

Try $i = .026$ $P = \$97.62$ (only \$0.80 off but not close enough)

Price is too low, so yield is still too high. Try $i = .024$ $P = \$99.20$ (\$0.79 off)

Try $i = .025$ $P = \$98.406$ close enough.

The annual yield to maturity is 5%.

Of course you may have hit on .025 much faster. That would be OK too.

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3. *[15 marks]*

Sophie has \$4.40 in nickels (5¢), dimes (10¢), and quarters (25¢). She has four times as many dimes as quarters. She has a total of 40 coins. How many of each coin does she have?
(Hint: Set up a linear system of equations and solve by using matrix reduction.)

Let N = # nickels
D = # dimes
Q = # quarters

$$5N + 10D + 25Q = 440 \quad (\text{or } .05N + .10D + .25Q = 4.40)$$
$$D - 4Q = 0 \quad (\text{or } D = 4Q)$$
$$N + D + Q = 40$$

$$\begin{array}{cccc} N & D & Q & \\ \left(\begin{array}{ccc|c} 1 & 1 & 1 & 40 \\ 0 & 1 & -4 & 0 \\ 5 & 10 & 25 & 440 \end{array}\right) \end{array}$$

$$\longrightarrow \left(\begin{array}{ccc|c} 1 & 1 & 1 & 40 \\ 0 & 1 & -4 & 0 \\ 0 & 5 & 20 & 240 \end{array}\right) \quad R_3 \to -5R_1 + R_3$$

$$\left(\begin{array}{ccc|c} 1 & 1 & 1 & 40 \\ 0 & 1 & -4 & 0 \\ 0 & 0 & 40 & 240 \end{array}\right) \quad R_3 \to -5R_2 + R_3$$

$$\begin{array}{cccc} N & D & Q & \\ \left(\begin{array}{ccc|c} 1 & 1 & 1 & 40 \\ 0 & 1 & -4 & 0 \\ 0 & 0 & 1 & 6 \end{array}\right) \end{array} \quad R_3 \to \tfrac{1}{6}R_3$$

back-substitution:

So $Q = 6$
$D = 4Q = 24$
$N = 40 - Q - D = 10$

or complete reduction:

$$\left(\begin{array}{ccc|c} 1 & 0 & 0 & 10 \\ 0 & 1 & 0 & 24 \\ 0 & 0 & 1 & 6 \end{array}\right)$$

$R_2 \to 4R_3 + R_2$

$R_1 \to -R_3 - R_2 + R_1$ (new R_3, R_2)

$N = 10$
$D = 24$
$Q = 6$

as before

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4. *[15 marks]*

Consider the system of linear equations

$$\begin{aligned} x_1 + 2x_2 + \ x_3 &= 0 \\ x_1 - \ x_2 + kx_3 &= 0 \\ x_1 + \ x_2 - \ x_3 &= 0 \end{aligned}$$

where k is some constant.

[9] (a) For what values(s) of k is there a solution other than the obvious $x_1 = x_2 = x_3 = 0$?

$$\left(\begin{array}{ccc|c} 1 & 2 & 1 & 0 \\ 1 & -1 & k & 0 \\ 1 & 1 & -1 & 0 \end{array}\right) \xrightarrow[R_3 \to -R_1+R_3]{R_2 \to -R_1+R_2} \left(\begin{array}{ccc|c} 1 & 2 & 1 & 0 \\ 0 & -3 & -k-1 & 0 \\ 0 & -1 & -2 & 0 \end{array}\right) \xrightarrow{R_2 \leftrightarrow R_3} \left(\begin{array}{ccc|c} 1 & 2 & 1 & 0 \\ 0 & -1 & -2 & 0 \\ 0 & -3 & k-1 & 0 \end{array}\right)$$

$$\xrightarrow[R_3 \to -3R_2+R_3]{R_2 \to -R_2} \left(\begin{array}{ccc|c} 1 & 2 & 1 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & k+5 & 0 \end{array}\right)$$

If $k+5 \neq 0$ there is a unique soln $x_1 = x_2 = x_3 = 0$.
To get other solutions $k+5 = 0$

$$\boxed{k = -5}$$

[6] (b) For the value(s) of k you found in (a), what is the most general solution to the system?

In case $k = -5$

$$\begin{array}{c} x_1 \ x_2 \ x_3 \\ \left(\begin{array}{ccc|c} 1 & 2 & 1 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right) \end{array}$$

back substitution

$x_2 = -2x_3$

$x_1 = -x_3 - 2x_2 = -x_3 + 4x_3 = 3x_3$

so $\boxed{\begin{array}{l} x_1 = 3x_3 \\ x_2 = -2x_3 \end{array}}$

or complete reduction

$$\begin{array}{c} x_1 \ x_2 \ x_3 \\ \xrightarrow{R_1 \to -2R_2+R_1} \left(\begin{array}{ccc|c} 1 & 0 & -3 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right) \end{array}$$

$x_1 = 3x_3$ as before
$x_2 = -2x_3$