Solved

A

Department of Mathematics
University of Toronto

Tuesday, October 28, 2014, 6:10-8:00 PM
MAT 133Y TERM TEST #1

Calculus and Linear Algebra for Commerce
Duration: 1 hour 50 minutes

Aids Allowed: A non-graphing calculator, with empty memory, to be supplied by student.

Instructions: Fill in the information on this page, and make sure your test booklet contains 10 pages. In addition, you should have a **multiple-choice answer sheet**, on which you should fill in your name, number, tutorial time, tutorial room, and tutor's name.

This test consists of 10 multiple choice questions, and 4 written-answer questions.
For the **multiple choice questions** you can do your rough work in the test booklet, but you must record your answer by circling the appropriate letter **on the answer sheet** with your pencil. Each correct answer is worth 4 marks; a question left blank, or an incorrect answer, or two answers for the same question is worth 0. For the **written-answer questions**, present your solutions in the space provided. The value of each written-answer question is indicated beside it.

ENCLOSE YOUR FINAL ANSWER IN A BOX AND WRITE IT IN INK.

TOTAL MARKS: 100

FAMILY NAME: ______________________

GIVEN NAME: ______________________

STUDENT NO: ______________________

SIGNATURE: ______________________

TUTORIAL TIME and ROOM: ______________________

REGCODE and TIMECODE: ______________________

T.A.'S NAME: ______________________

Regcode	Timecode	Room	**Regcode**	Timecode	Room
T0101A	M9A	SS1086	T0501B	W3B	UC261
T0101B	M9B	SS1088	T0601A	R4A	BA2195
T0101C	M9C	SS2128	T0601B	R4B	UC216
T0201A	M3A	UC261	T0601C	R4C	UC330
T0201B	M3B	BA2145	T0601D	R4D	BL112
T0201C	M3C	BAB024	T0701A	F2A	SS1070
T0201D	M3D	BF323	T0701B	F2B	SS1074
T0301A	T3A	UC261	T0701C	F2C	SS2127
T0301B	T3B	MS2173	T0701D	F2D	SS2106
T0301C	T3C	WW126	T0801A	F3A	SS1070
T0301D	T3D	IN204	T0801B	F3B	SS1074
T0401A	W9A	LM155	T5101A	M5A	BA1210
T0401B	W9B	LM157	T5101B	M5B	BL114
T0401C	W9C	MP118	T5201A	M6A	SS1088
T0501A	W3A	SS1070			

FOR MARKER ONLY	
Multiple Choice	
B1	
B2	
B3	
B4	
TOTAL	

NAME: ______________________ STUDENT NO: ________________

PART A. Multiple Choice

1. *[4 marks]*

If a \$100 account with interest compounded annually grows to \$119.03 in 5 years, what is its nominal annual interest rate (to the nearest 0.01%)?

A. 3.40%

B. 3.55%

C. 3.45%

D. 3.50%

E. 3.60%

Let r = nominal annual interest rate.

$119.03 = 100(1+r)^5$

$r = \left(\frac{119.03}{100}\right)^{\frac{1}{5}} - 1$

$= .035\underline{455}\ldots \quad 455 \to 5$

3.55% B

2. *[4 marks]*

A 30-year loan with interest at 6% per year compounded monthly has monthly payments of \$2098.43. The value of the loan is closest to:

A. \$353, 000

B. \$325, 000

C. \$375, 000

D. \$350, 000

E. \$300, 000

i = monthly interest rate

$i = .005$

$A = 2098.43\, a_{\overline{360}|.005}$

$= 2098.43 \left[\frac{1-(1.005)^{-360}}{.005}\right]$

$= 350,000.53$

D

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3. *[4 marks]*

A fund is set up in order to donate \$10,000 to a certain charity at the beginning of each year, in perpetuity. If the first donation is to be 1 year after the fund is set up and interest is 7% compounded annually, how much money (to the nearest dollar) should be used to set the fund up?

A. \$140,400

B. \$144,293

C. \$137,062

D. \$138,925

E. \$142,857

1 n

10,000 10,000

In a perpetuity, if A is the amount and R is the periodic payment, and r is the rate per period

$R = rA$

and $A = \frac{R}{r} = \frac{10,000}{.07}$

$= 142,857.14$

E

or $A = \lim_{n\to\infty} R\, a_{\overline{n}|r}$

$= \lim_{n\to\infty} R\left[\frac{1-(1+r)^{-n}}{r}\right] = \frac{R}{r}$ as above

4. *[4 marks]*

A person opens an account at the beginning of January, 2014 with a deposit of \$100 and continues making deposits of \$100 each at the beginning of each month of 2014, with the last deposit at the beginning of December, 2014. If interest is 6% compounded monthly, then at the beginning of January, 2015, the account will have

A. \$1233.56

B. \$1237.41

C. \$1239.72

D. \$1228.75

E. \$1225.24

2014 Jan 1 2014 Dec 1 2015 Jan 1

-1 0 1 11 12

100 100

$r = .005$

This is an annuity due. There are many ways to get the value on Jan 1 2015.

One way:

On Dec 1, 2014 the account will have

$100\, s_{\overline{12}|.005}$

So one month later

$1.005 \times 100\, s_{\overline{12}|.005}$

$\approx \$1239.72$ C

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5. *[4 marks]*

A family has a $400,000 mortgage amortized over 15 years with monthly payments (at the end of each month) at an interest rate of 3% per year compounded semi-annually. The first payment is at the end of the first month. The interest in the first payment is closest to

$(1+i)^{12} = (1.015)^2$

A. $ 993.81

B. $1000.00

C. $1762.33

D. $1768.53

E. zero because there is no interest in the first payment

At the beginning of the 1st period, the principal outstanding is \$400,000, so the interest in the 1st payment is

$400{,}000 \times i = 400{,}000\left[(1.015)^{\frac{1}{6}} - 1\right]$

$\approx \$993.81$ A

6. *[4 marks]*

A $1000 loan is to be repaid by 2 equal payments: the first payment 1 year from now and the second payment 2 years from now. If interest is 5% compounded continuously, what is the amount of each payment?

A. $545.22

B. $532.35

C. $542.09

D. $548.40

E. $538.77

0	1	2
1000	X	X

$1000 = Xe^{-.05} + Xe^{-2(.05)}$

$X = \dfrac{1000}{e^{-.05} + e^{-.1}}$

$\approx \$538.77$ E

A

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7. *[4 marks]*

Let $A = \begin{bmatrix} 7 & 0 & 4 \\ 2 & 3 & -1 \\ 1 & 5 & 0 \end{bmatrix}$, $B = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 2 & 1 \\ 1 & 4 & -1 \end{bmatrix}$ and I be the 3×3 identity matrix. If $C = A^T B - 3I$, then c_{13} is

A. -2

B. 8

C. 0

D. 3

E. 23

$$A^T B - 3I = \begin{pmatrix} 7 & 2 & 1 \\ 0 & 3 & 5 \\ 4 & -1 & 0 \end{pmatrix}\begin{pmatrix} 0 & 0 & 1 \\ 0 & 2 & 1 \\ 1 & 4 & -1 \end{pmatrix} - \begin{pmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 8 & 8 \\ 5 & 26 & -2 \\ 0 & -2 & 3 \end{pmatrix} - \begin{pmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{pmatrix}$$

$$= \begin{pmatrix} -2 & 8 & 8 \\ 5 & 23 & -2 \\ 0 & -2 & 0 \end{pmatrix}$$

$C_{13} = 8$

B

8. *[4 marks]*

If A is a 1×3 matrix, B is a 2×3 matrix and C is a 3×1 matrix then the dimensions of the matrix given by

$$B(A^T + C) + BA^TAC$$

$2 \times 1 + 2 \times 1$

$= 2 \times 1$

are

A. 1×3

B. 2×3

C. 3×1

D. 2×1

E. 3×2

D

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9. *[4 marks]*

Which of the following augmented coefficient matrices represents a system of equations which has no solution?

A. $\left[\begin{array}{ccc|c} 1 & 2 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right]$

B. $\left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right]$

C. $\left[\begin{array}{cc|c} 1 & 5 & 3 \\ 0 & 0 & 4 \end{array}\right]$

D. $\left[\begin{array}{ccccc|c} 1 & 9 & 0 & 0 & 0 & 5 \\ 0 & 0 & 1 & 0 & 0 & 6 \\ 0 & 0 & 0 & 1 & 1 & -3 \end{array}\right]$

E. $\left[\begin{array}{ccc|c} 1 & 7 & 8 & 3 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$

All matrices are in row-echelon form, and only C has a row of zeros with a non-zero right-hand-side

C

10. *[4 marks]*

The augmented coefficient matrix of a system of equations reduces to

$$\left[\begin{array}{ccccc|c} 1 & 0 & 0 & 1 & 0 & 3 \\ 0 & 1 & 0 & 0 & 1 & 2 \\ 0 & 0 & 1 & 0 & 0 & -1 \end{array}\right]$$

The system has:

There are 5 variables and 3 equations remain: therefore, there are 2 parameters

D

A. one unique solution

B. no possible solutions

C. a 1-parameter family of solutions

D. a 2-parameter family of solutions

E. a 3-parameter family of solutions

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PART B. Written-Answer Questions

1. *[13 marks]*

Teresa has \$700 in a bank account earning an effective annual rate of 3%. In 10 years time, she will take all the money in the account and purchase a \$1000 Government Bond that has been on the market for some time.

[4] (a) How much is the bond selling for if she uses all of the money in the account to buy it?

$$700(1.03)^{10} = \boxed{\$940.74}$$

[9] (b) At the time that she purchases the bond the following is true:

i) the annual yield rate is 3%

ii) the bond has semi-annual interest payments and the next one is due in 6 months

iii) the bond matures in 6 more years

What is the annual coupon rate of the bond?

$i = .015$ $\qquad r =$ semi-annual coupon rate

$n = 12$

$V = 1000$

$$940.74 = 1000(1.015)^{-12} + r * 1000\, a_{\overline{12}|.015}$$

Solve for r.

$$r \approx .009567044$$

$$2r \approx .01913$$

Annual coupon rate $\boxed{1.91\%}$

A

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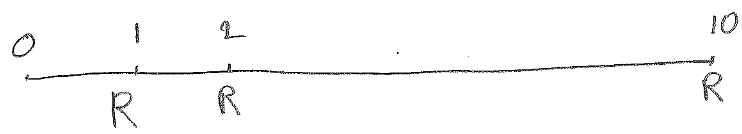
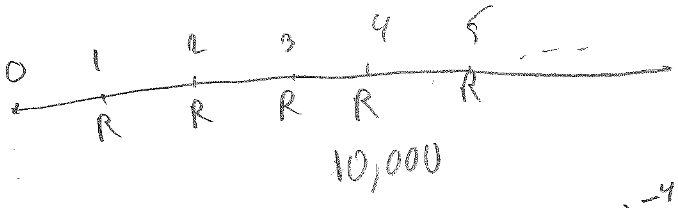
2. *[13 marks]*

A \$100,000 loan is amortized over 10 years at 4% per year with interest compounded annually and annual payments at the end of each year. If an extra payment of \$10,000 (on top of the regular payment) is made at the end of the fourth year, how many payments are there altogether until the loan is paid off?



$$100{,}000 = R\, a_{\overline{10}|.04}$$

$$R = \frac{100{,}000}{a_{\overline{10}|.04}} = $$ (\$12,329.09) though we don't need to know this.



$$100{,}000 = 10{,}000(1.04)^{-4} + R\, a_{\overline{n}|.04}$$ where n is the number of payments

$$100{,}000 = 10{,}000(1.04)^{-4} + \frac{100{,}000}{a_{\overline{10}|.04}}\, a_{\overline{n}|.04}$$

$$100{,}000 = 10{,}000(1.04)^{-4} + 100{,}000\,\frac{[1-(1.04)^{-n}]}{[1-(1.04)^{-10}]}$$

Solve for n: e.g. divide by 100,000

$$1 = \frac{(1.04)^{-4}}{10} + \frac{[1-(1.04)^{-n}]}{[1-(1.04)^{-10}]}$$

$$\left[1 - \frac{(1.04)^{-4}}{10}\right]\left[1-(1.04)^{-10}\right] = \left[1-(1.04)^{-n}\right]$$

$$(1.04)^{-n} = \left\{1 - \left[1 - \frac{(1.04)^{-4}}{10}\right]\left[1-(1.04)^{-10}\right]\right\}$$

$$n = -\frac{\ln\left\{\frac{(1.04)^{-4}}{10} + (1.04)^{-10} - \frac{(1.04)^{-14}}{10}\right\}}{\ln(1.04)}$$

$n \approx 8.97423\ldots$ So 8 full payments + 1 smaller payment

$\boxed{\text{9 payments in all}}$

A

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3. *[17 marks]*

[10] (a) Find all the solutions to the following system of equations using row reduction. [No marks for any other method.]

$$\begin{aligned} 2x_2 + 6x_3 \phantom{{}+3x_4} &= 1 \\ x_1 \qquad + 2x_3 + 3x_4 &= -1 \\ 3x_1 \qquad + 4x_3 + x_4 &= 5 \end{aligned}$$

$$\begin{array}{cccc} x_1 & x_2 & x_3 & x_4 \end{array}$$
$$\left(\begin{array}{cccc|c} 0 & 2 & 6 & 0 & 1 \\ 1 & 0 & 2 & 3 & -1 \\ 3 & 0 & 4 & 1 & 5 \end{array}\right) \xrightarrow[R_3 \to -3R_1 + R_3]{R_1 \leftrightarrow R_2} \left(\begin{array}{cccc|c} 1 & 0 & 2 & 3 & -1 \\ 0 & 2 & 6 & 0 & 1 \\ 0 & 0 & -2 & -8 & 8 \end{array}\right) \xrightarrow[R_3 \to -\frac{1}{2}R_3]{R_2 \to \frac{1}{2}R_2} \begin{array}{c} \begin{array}{cccc} x_1 & x_2 & x_3 & x_4 \end{array} \\ \left(\begin{array}{cccc|c} 1 & 0 & 2 & 3 & -1 \\ 0 & 1 & 3 & 0 & \frac{1}{2} \\ 0 & 0 & 1 & 4 & -4 \end{array}\right) \end{array}$$

row-echelon form

back substitution: $x_3 = -4 - 4x_4$; $x_2 = \frac{1}{2} - 3x_3 = \frac{1}{2} - 3(-4 - 4x_4) = \frac{25}{2} + 12x_4$;

$x_1 = -1 - 3x_4 - 2x_3 = -1 - 3x_4 - 2(-4 - 4x_4) = 7 + 5x_4$

$$\boxed{\begin{aligned} x_1 &= 7 + 5x_4 \\ x_2 &= \tfrac{25}{2} + 12x_4 \\ x_3 &= -4 - 4x_4 \end{aligned}}$$

a 1-parameter family of solns.

or by complete reduction:

$R_2 \to -3R_3 + R_2$

$R_1 \to -2R_3 + R_1$

$$\begin{array}{cccc} x_1 & x_2 & x_3 & x_4 \end{array}$$
$$\left(\begin{array}{cccc|c} 1 & 0 & 0 & -5 & 7 \\ 0 & 1 & 0 & -12 & 12\frac{1}{2} \\ 0 & 0 & 1 & 4 & -4 \end{array}\right)$$

$$\boxed{\begin{aligned} x_1 &= 7 + 5x_4 \\ x_2 &= 12\tfrac{1}{2} + 12x_4 \\ x_3 &= -4 - 4x_4 \end{aligned}}$$

the same solution

[7] (b) Is there a solution to the system of equations from part (a) in which $x_1 = 2$ and $x_2 = 1$? Using your solution in part (a), either find this solution or explain why there is none.

If $x_2 = 1$, then the parameter $x_4 = -1$.

But then $x_2 = \frac{1}{2}$ not 1; so this is impossible.

$\boxed{\text{There is no such solution.}}$

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4. *[17 marks]*

[5] (a) For which values of k does the matrix $\begin{bmatrix} 1 & 2 \\ -2 & k \end{bmatrix}$ have an inverse, and what is A^{-1}? [Your answer will involve k.]

$$\left(\begin{array}{cc|cc} 1 & 2 & 1 & 0 \\ -2 & k & 0 & 1 \end{array}\right) \xrightarrow[R_2 \to 2R_1 + R_2]{} \left(\begin{array}{cc|cc} 1 & 2 & 1 & 0 \\ 0 & k+4 & 2 & 1 \end{array}\right) \xrightarrow[\substack{R_2 \to \frac{1}{k+4}R_2 \\ \text{only if } k \neq -4 \\ R_1 \to -2R_2 + R_1}]{} \left(\begin{array}{cc|cc} 1 & 0 & 1-\frac{4}{k+4} & -\frac{2}{k+4} \\ 0 & 1 & \frac{2}{k+4} & \frac{1}{k+4} \end{array}\right)$$

$$= \left(\begin{array}{cc|cc} 1 & 0 & \frac{k}{k+4} & -\frac{2}{k+4} \\ 0 & 1 & \frac{2}{k+4} & \frac{1}{k+4} \end{array}\right)$$

$\boxed{\text{If } k = -4 \text{, there is no } A^{-1}.}$

If $k \neq -4$, $A^{-1} = \boxed{\frac{1}{k+4}\begin{pmatrix} k & -2 \\ 2 & 1 \end{pmatrix}}$

[4] (b) Let $A = \begin{bmatrix} 1 & 2 \\ -2 & -3 \end{bmatrix}$. Use your solution to part (a) to solve $Ax = B$ where $B = \begin{bmatrix} 4 \\ -1 \end{bmatrix}$. [No marks if you don't use your solution in (a)].

Note that with $k = -3$ in (a) we have our A.

So $A^{-1} = \frac{1}{-3+4}\begin{pmatrix} -3 & -2 \\ 2 & 1 \end{pmatrix} = \begin{pmatrix} -3 & -2 \\ 2 & 1 \end{pmatrix}$. $Ax = B$ so $x = A^{-1}B$

$$x = \begin{pmatrix} -3 & -2 \\ 2 & 1 \end{pmatrix}\begin{pmatrix} 4 \\ -1 \end{pmatrix} = \begin{pmatrix} -12+2 \\ 8-1 \end{pmatrix} = \boxed{\begin{pmatrix} -10 \\ 7 \end{pmatrix}}$$

[8] (c) Two stores sell the same quantity of product A. The two stores also sell the same quantity of product B. The first store sells product A at \$1 above cost (per unit) and product B at \$2 per unit above cost. The second store sells product A at \$2 below cost (per unit) and product B at \$3 below cost. If the first store earns a total of \$100 and the second store loses \$150, how many units of each product did they sell? (**Hint: use your solution to part (a) or (b).**)

Let x_A = no. of units of product A sold by each store

x_B = no. " " " B " " "

Store 1 earns $x_A + 2x_B = 100$

Store 2 earns $-2x_A - 3x_B = -150$

$$\begin{pmatrix} 1 & 2 \\ -2 & -3 \end{pmatrix}\begin{pmatrix} x_A \\ x_B \end{pmatrix} = \begin{pmatrix} 100 \\ -150 \end{pmatrix}$$

Notice: $A\begin{pmatrix} x_A \\ x_B \end{pmatrix} = \begin{pmatrix} 100 \\ -150 \end{pmatrix}$ where A is the matrix in (b)

so $\begin{pmatrix} x_A \\ x_B \end{pmatrix} = A^{-1}\begin{pmatrix} 100 \\ -150 \end{pmatrix} = \begin{pmatrix} -3 & -2 \\ 2 & 1 \end{pmatrix}\begin{pmatrix} 100 \\ -150 \end{pmatrix}$. A^{-1} from (b)

$$\begin{pmatrix} x_A \\ x_B \end{pmatrix} = \begin{pmatrix} -300+300 \\ 200-150 \end{pmatrix} = \begin{pmatrix} 0 \\ 50 \end{pmatrix}.$$

$\boxed{\text{Each store sold none of product A and 50 units of product B.}}$